Iterative Fractals

Oliver Linton

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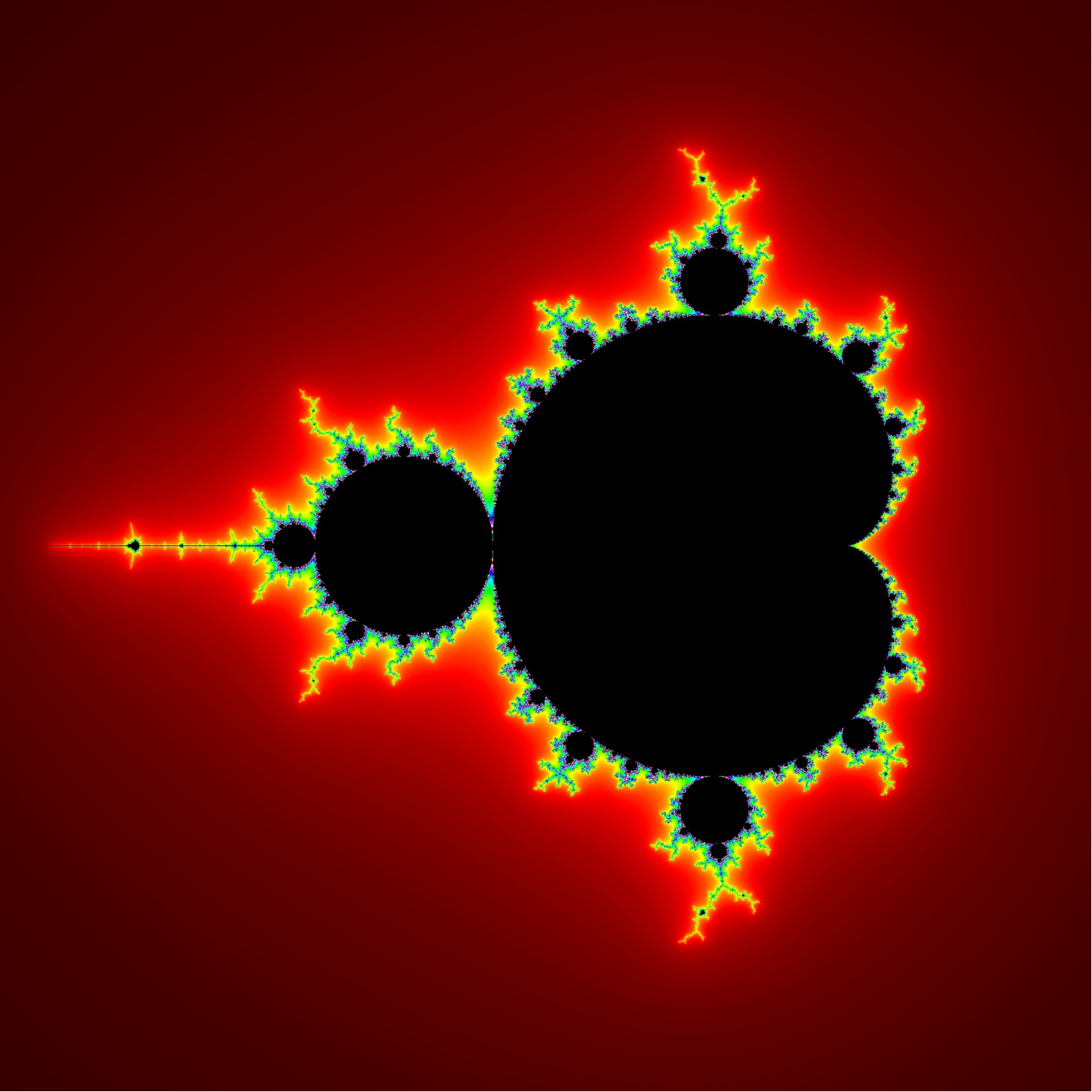


Introduction

There can be few people who have not heard of the famous Mandelbrot Set and seen fabulous images of its wonders in books and on the web. I myself have already written a number of books on the subject which explain in some detail how the fractal and others like it are generated. In this book I wish to explore some different ways in which the basic algorithm can be altered to produce a whole new range of fractal images.

First, let us review the process by which the basic image, shown opposite, is created.

Any point on the plane can be specified by quoting its X and Y coordinates. Often we write these as a pair of bracketed numbers; so, for example, (5, -3) represents a point 5 units to the right of the origin and 3 units below it.

There is, however, a much more powerful way of dealing with points on the plane using what are known as complex numbers. The same point is represented by the complex number $5 - 3i$ where i is the square root of minus one. For our purposes, it is not necessary to understand how complex numbers work – suffice it to say that they can be added and multiplied just like ordinary numbers and that the result is always another complex number which can, of course, represent a new point in the complex plane.

Now the standard method of generating a fractal like the image opposite is to start with two complex numbers z and C. Perform a simple mathematical operation on z and C to get a new complex number z'. Replace z with z' and repeat the operation over and over again. Note that C remains constant throughout this process but z will dot around all over the place.

Sometimes, z will get trapped in some kind of cycle. More often it will escape to infinity. It is usual to specify a circle round the origin at a suitable distance and count how many steps are needed for z to escape beyond this limit. If, after a certain number of iterations, z is still within the limit set, it is deemed to be stable. This number is then used to colour a point on the screen in one of two ways:

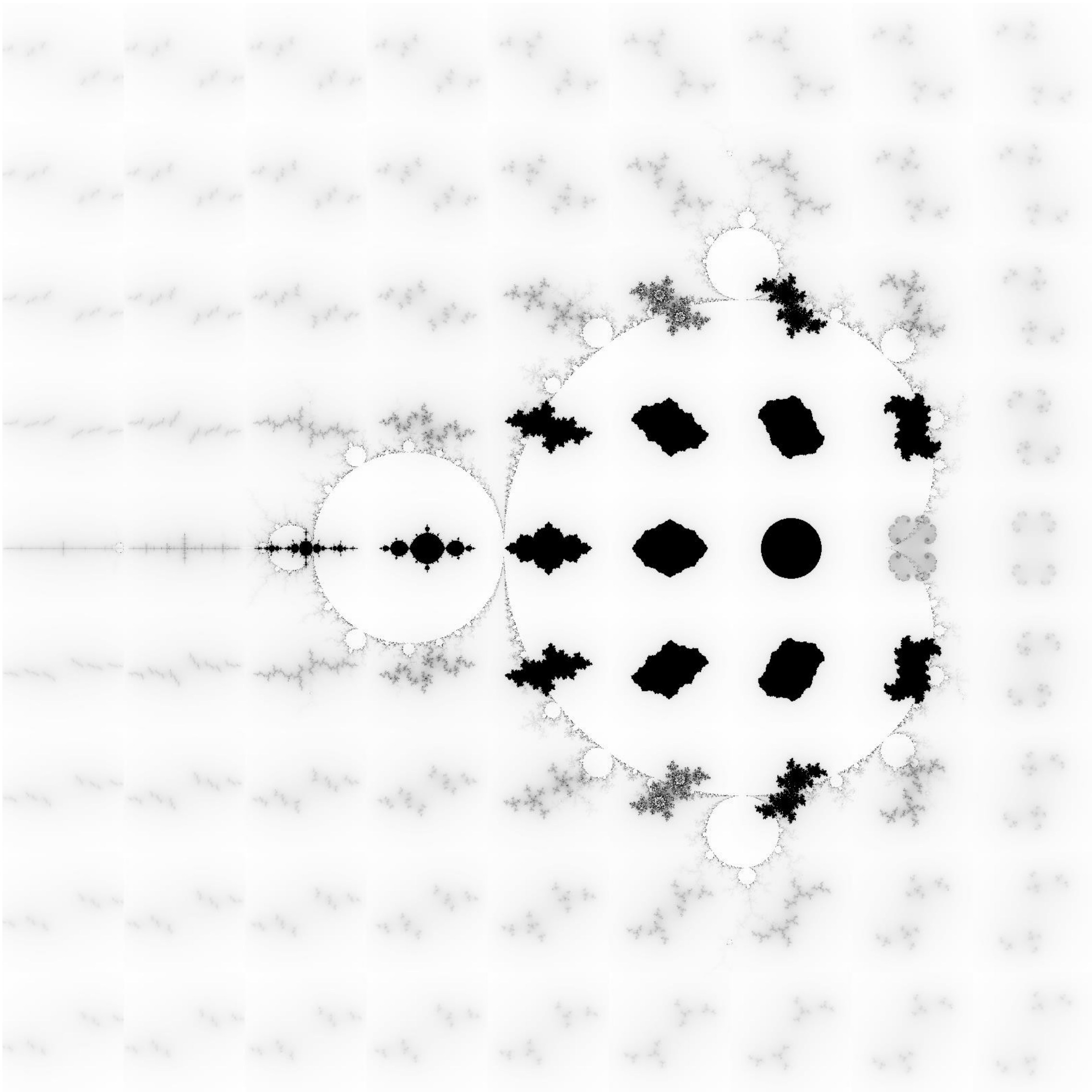
If you colour in the point represented by the initial starting position z_0 you will get what is known as the *Julia Set* for the value C. What this means is that for any given algorithm or mathematical operation, there is a different fractal image for every value of C.

On the other hand, if you fix the starting point z_0 (usually at (0, 0)) and colour the point on the screen according to different values of C you will obtain the famous *Mandelbrot Set*. In fact you can think of the Mandelbrot Set as a kind of map of all the possible Julia sets.

Stable points are usually printed as black. I call this the 'lake' colour.

The illustration opposite shows small images of the Julia sets superimposed on a a larger image of the Mandelbrot set. You will notice that the Julia sets whose value of C lies outside the main blob of the Mandelbrot set are fragmentary, but those whose C lies inside the Mandelbrot set have large stable areas. Julia sets whose C lies close to the boundary of the Mandelbrot set can have very intricate structures.

It is also of interest to note that the fractal structure of the Mandelbrot set at any point near the boundary bears a remarkable resemblance to the Julia set which corresponds to that point.



The Teddy Bear Sets

The algorithm for the standard Mandelbrot Set is as follows:

$$z' = z^2 + C$$

This produces a stable cardioid with a whole series of 'lobes' attached to it as illustrated in the frontispiece.

The most obvious thing to try is equations of the form

$$z' = z^n + C$$

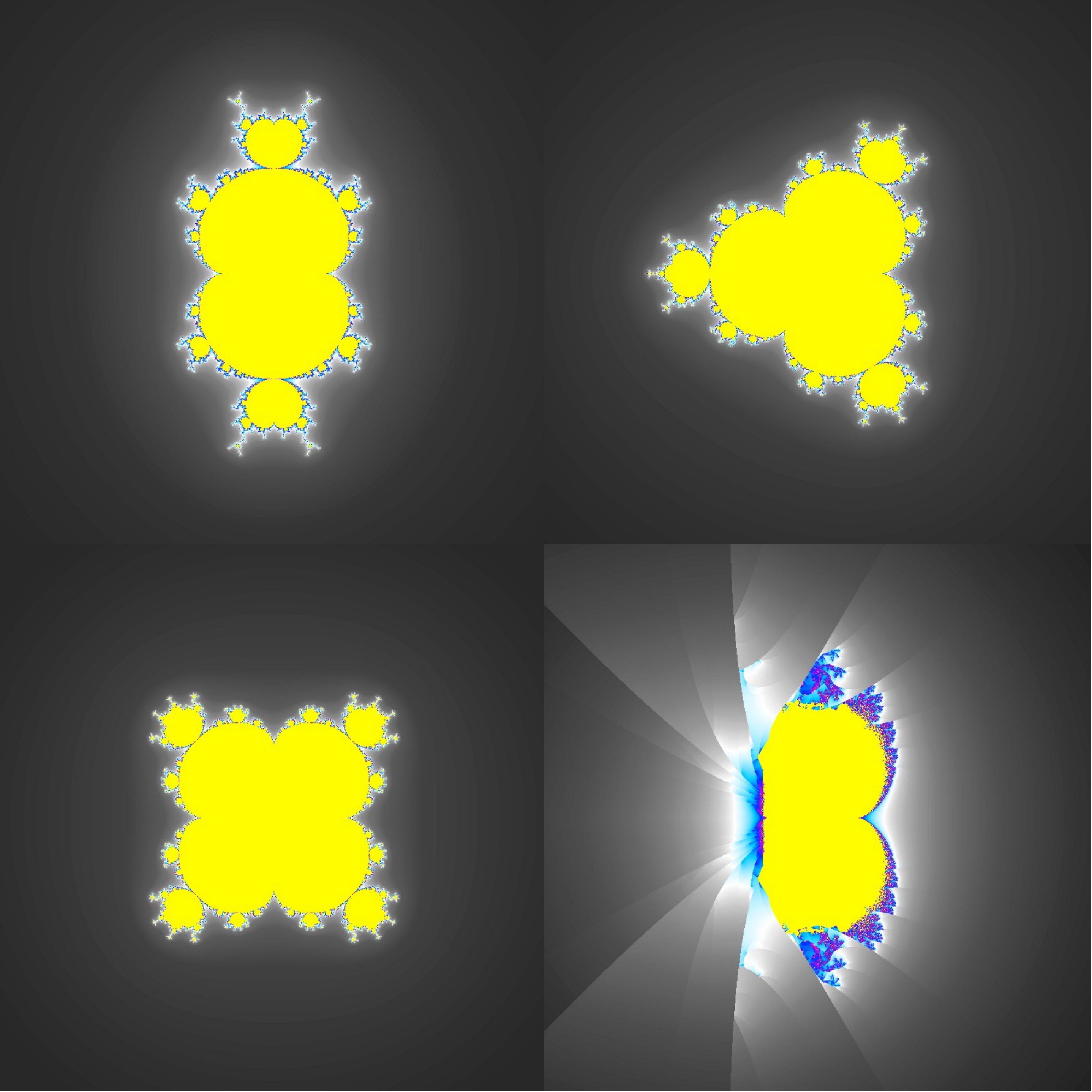
where $n = 3, 4, 5$ etc. The results are shown on the opposite page. I call them 'Teddy Bear' sets because they sprout 'ears' on the lobes.

The image in the lower right corner is obtained with a fractional value of n = 1.5. i.e:

$$z' = \sqrt{z}^{\,3} + C$$

What this means is that at some stage in the calculation it is necessary to take the square root of z. Now every number has two square roots and the program must decide which to adopt. This is fine most of the time but as z circles round the origin, there comes a point where the program has to switch from one root to the other. This causes a discontinuity in the image and destroys its aesthetic integrity.

(Note that I shall refer to all fractals generated by iterating with different values of C as 'Mandelbrot' sets to differentiate them from 'Julia' sets which are iterated with a constant value of C and different initial points z_0.)



Generalised polynomials

Next we shall extend our range of polynomial functions by adding extra terms.

It turns out that the function

$$z' = z^2 - Rz + C$$

is not all that interesting. The extra term simply moves the standard Mandelbrot set to a new position in the plane. The function

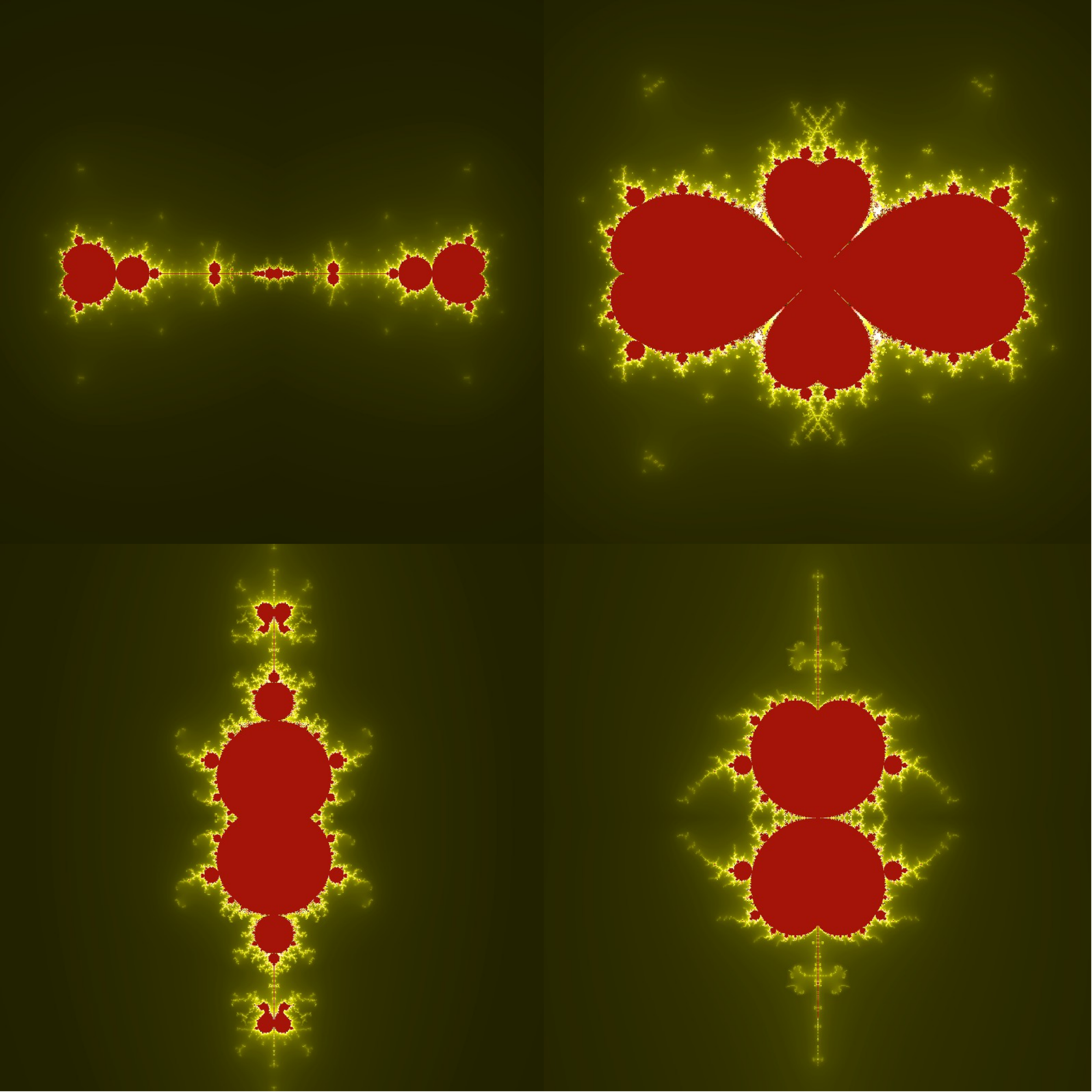
$$z' = z^3 - Rz + C$$

turns up some surprises however.

The images opposite are the Mandelbrot sets generated when $R = 2$ and 1 respectively on the top row and –1 and –2 along the bottom. (Remember that when $R = 0$ the set will be the Teddy Bear set for $n = 3$.)

Cubic equations like this one have, in general, two critical points – that is to say, there are two starting points which result in a coherent fractal. The critical points for this function are $\pm\sqrt{R/3}$.

The images opposite are a composite of the fractals produced by both critical points. The points which are coloured red represent values of C which result in a stable orbit for at least one of the critical starting points.



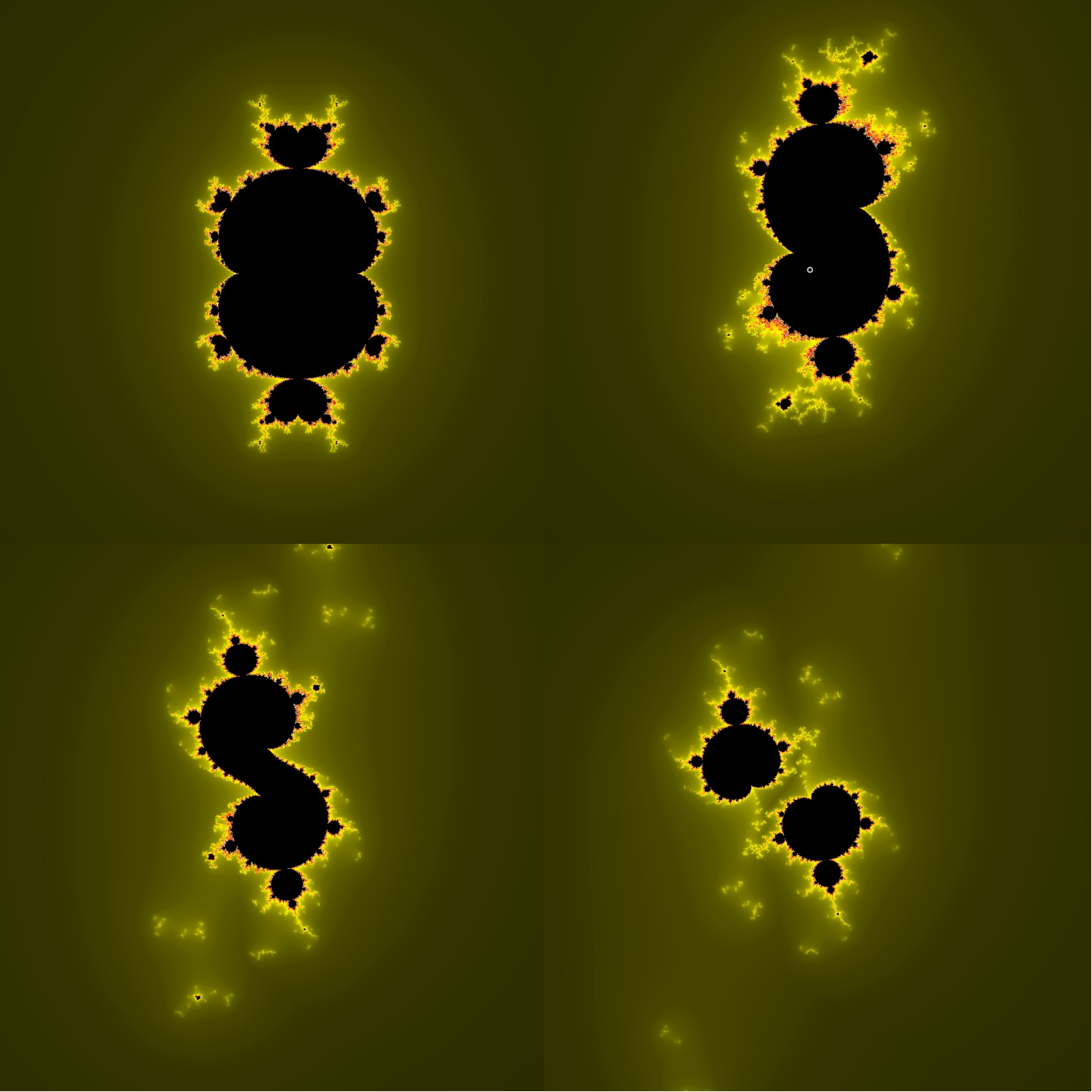
Symmetry

It will not have escaped your notice that all the Mandelbrot sets illustrated so far are symmetrical about the horizontal (real) axis; others are symmetrical about the vertical (imaginary) axis too and many have rotational symmetry. Why is this?

The vertical symmetry is characteristic of polynomials which contain only odd terms such as z and z^3.

The horizontal symmetry is due to an entirely different cause. It is due to a fundamental property of complex numbers. Every complex number $z = x + iy$ has a counterpart on the opposite side to the real axis known as its complex conjugate $\bar{z} = x - iy$ It is easy to show that when you add or multiply the complex conjugates of two complex numbers, the result is simply the complex conjugate of the original result. What this means is that whatever formula you apply to z using a certain value of C, z will perform an exact mirror image when you use the complex conjugate of C. The same is true if you multiply a complex number by a real number. But the feature breaks down if you multiply a complex number by a constant which has an imaginary component.

Consider the function: $z' = z^3 - R z^2 + C$ in which the constant R is a complex number. In the illustrations opposite R has the values equal 0.5, 0.5 + i, 0.5 + 1.5i and 0.5 + 2i. Not only is the horizontal and vertical symmetry destroyed, the centre of rotational symmetry is not even centred on the origin (indicated by a circle in the second image).



Logistic Functions

An important function widely used to predict population growth is

$$x' = A\,x\,(1 - x)$$

Its complex equivalent is

$$z' = C\,z\,(1 - z)$$

This function differs from those we have used so far in that the vital constant C is used to multiply the polynomial rather than being added. The effect of this is to render the fractal symmetric about both the X and Y axes.

The quadratic version quoted above generates a fractal with two large circular lobes (see the upper illustration opposite) whereas the following equation

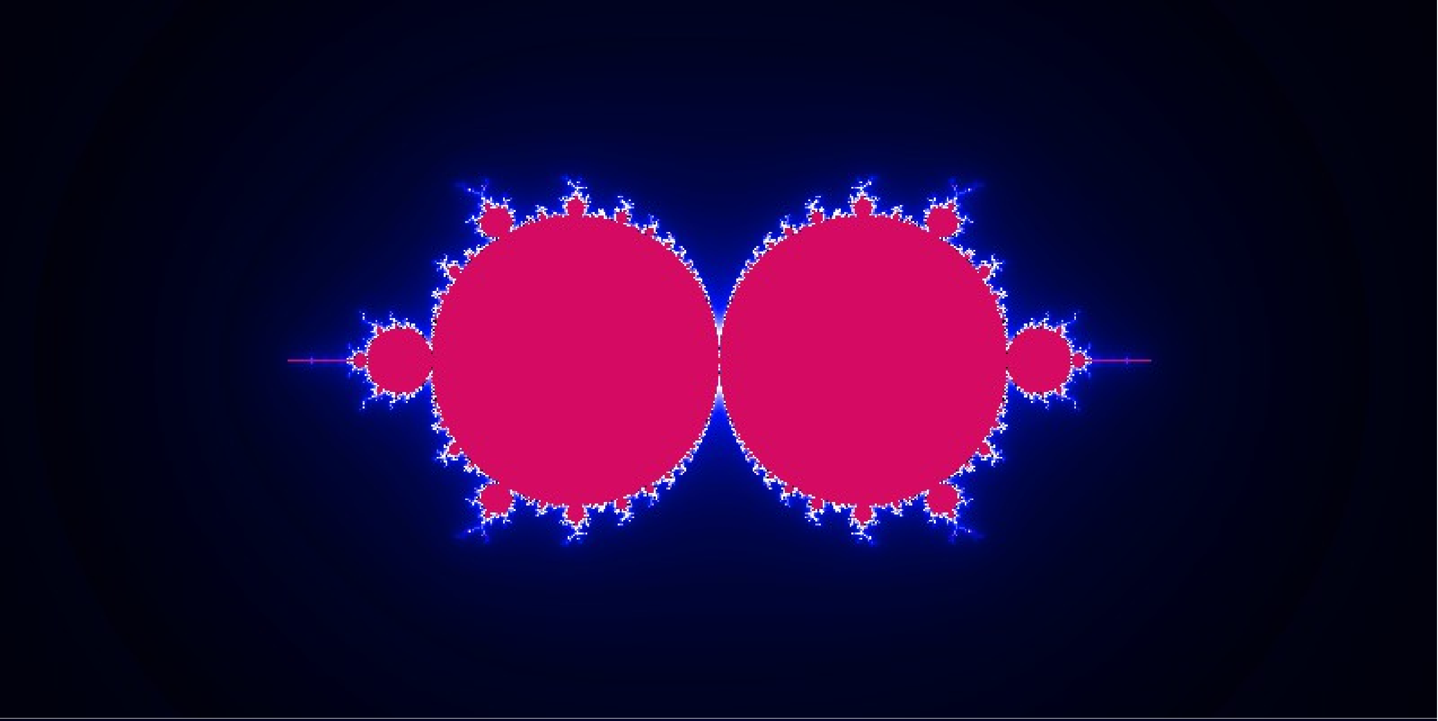
$$z' = C\,z\,(1 - z^2)$$

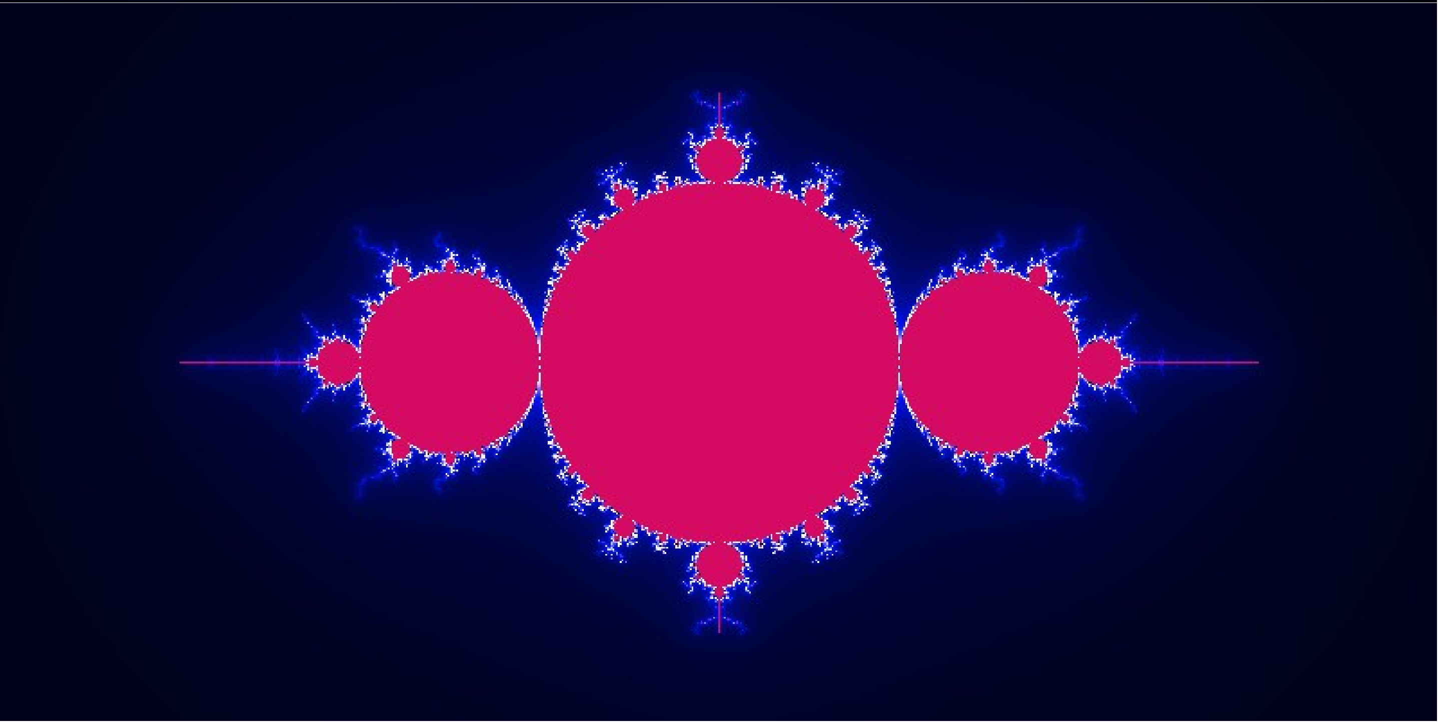
(which is a cubic equation) has one large lobe flanked by two smaller ones.

I will invite you to guess what the Mandelbrot set for the quartic version

$$z' = C\,z\,(1 - z^3)$$

will look like.





Inverted Fractals

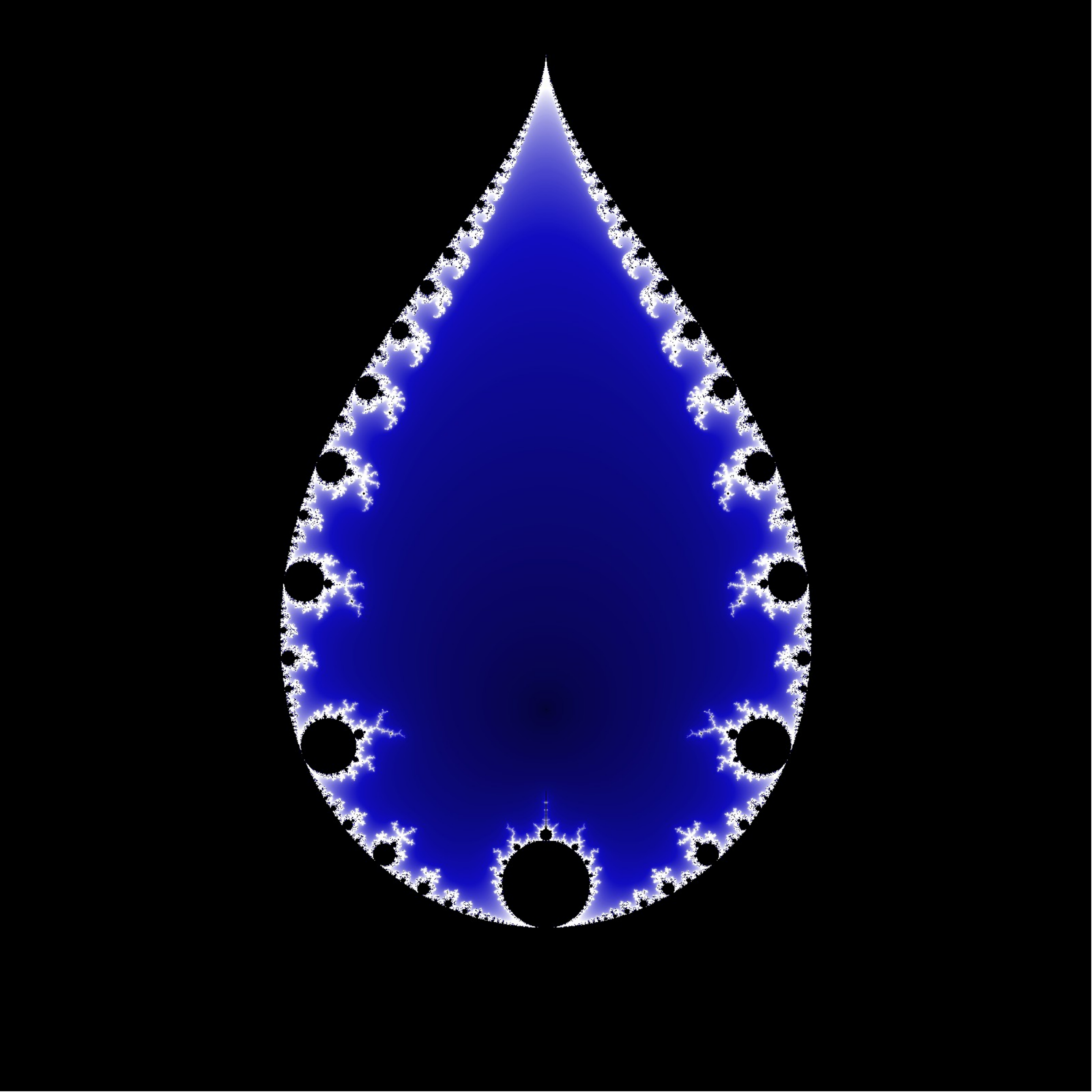
All the examples we have seen so far are essentially the same – oddly shaped islands of stability with various lobes and tendrils attached. All the interesting activity lies outside the island of stability.

We can often turn a mundane image into a thing of beauty by turning it inside out so that the majority of the plane is stable and the lobes and filaments point inwards. This is easily achieved by using $1/C$ instead of C. The generating algorithm for the standard Mandelbrot set therefore becomes:

$$z' = z^2 + 1/C$$

and what it produces is a complete surprise. (The image opposite has been rotated by 90 degrees anti-clockwise to make it look like a pendant jewel.)

The cusp of the cardioid has been transformed into the tip of the tear drop and all the lobes round the perimeter of the cardioid have been turned inwards.



Celtic Gold

I said earlier that the generalised quadratic function

$$z' = z^2 - R\,z + C$$

is not all that interesting. That is true – but its inverse

$$z' = z^2 - R\,z + 1/C$$

is extremely interesting.

As we have seen, when $R = 0$ the result is the tear drop pendant.

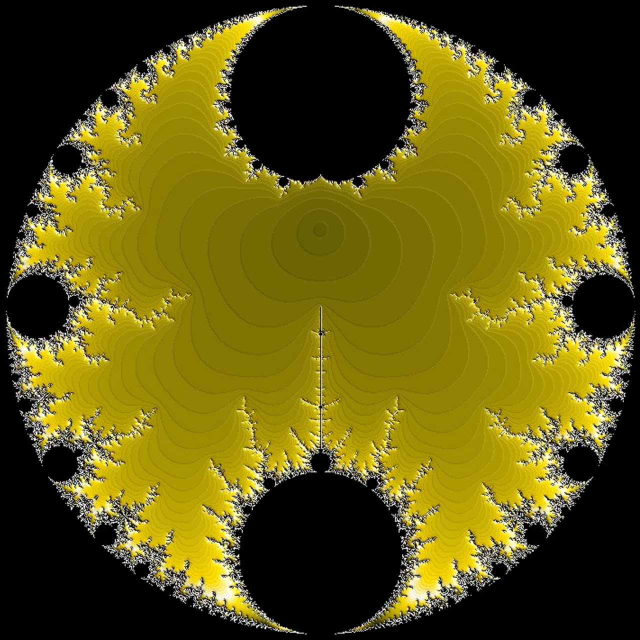
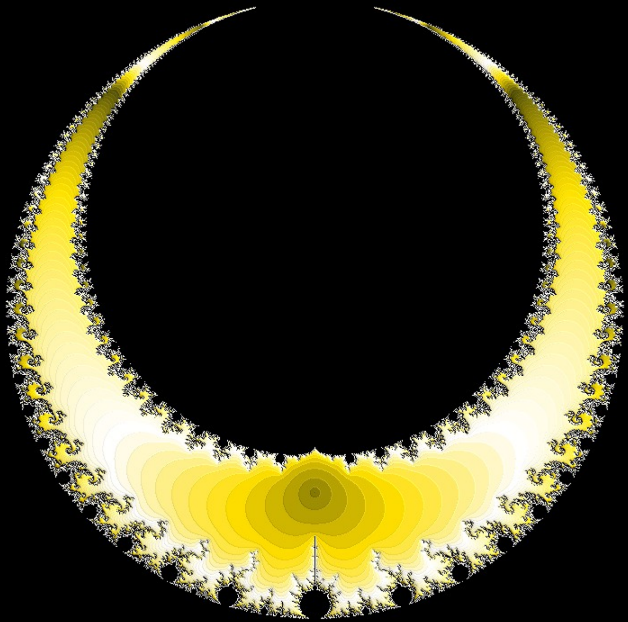
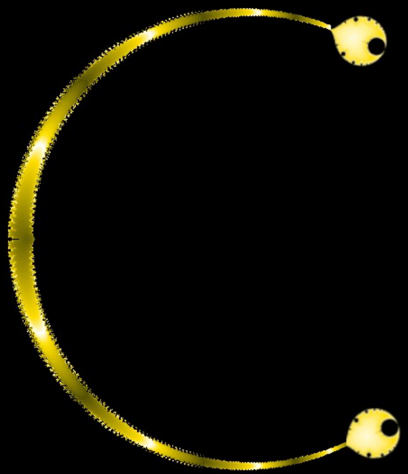
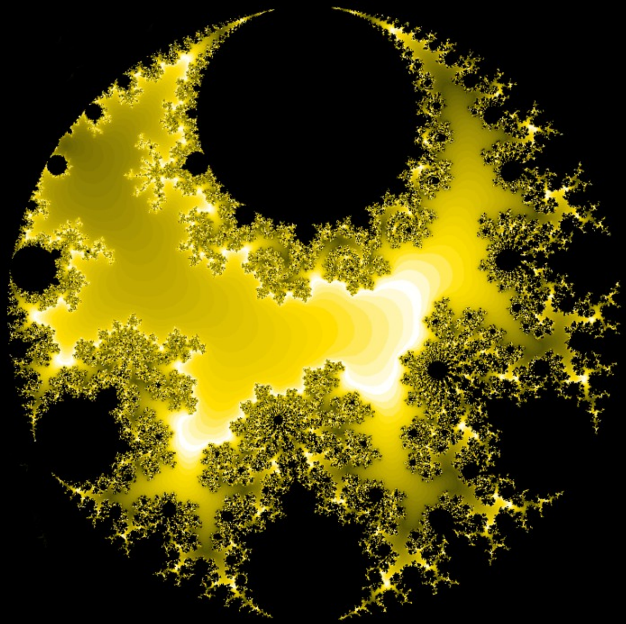
Other values of R produce the results shown opposite. At the top left, $R = 1.3$ we could have a design for a Celtic shield with strict left-right symmetry.

Next to it with $R = 1.1$ we obtain a Celtic lunula – worn by a wealthy lady round the neck

The image at the bottom left resembling a Celtic torc is a composite of three values of R the main one being 1.02.

The last has R equal to the complex value of $0.9 - 0.4i$ and could be a chieftain's breastplate.

(Note that these images have been rotated and embossed for effect.)

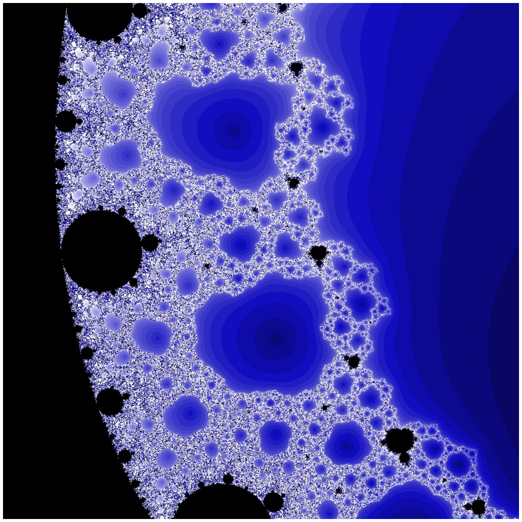





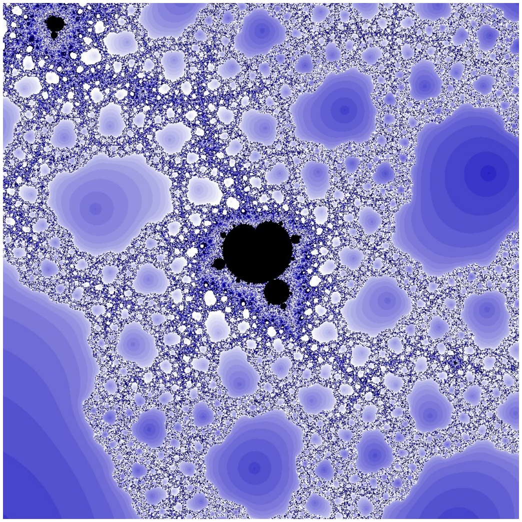
A Filligree Pendant

This gorgeous fractal is generated by iterating another equation containing a reciprocal term:

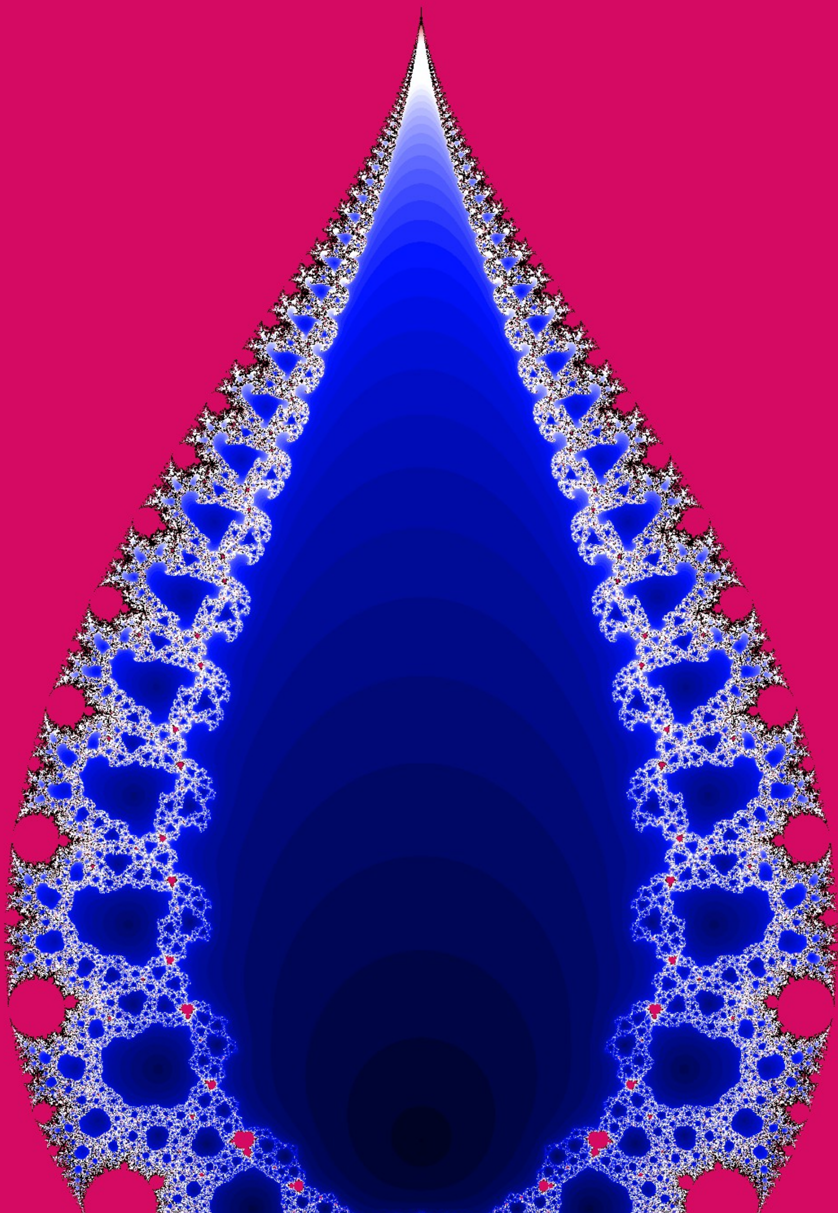
$$z' = (z^2 + 1/z)/C$$

I do not know what causes the formula to produce such delicate lacework but, as usual, you can find minibrot holes in it everywhere. (Note that, like the tear drop on page 11, this is an inverted fractal. It has also been shifted slightly in the X direction before inversion. The effect of this is to change the overall shape of the fractal dramatically.)





More reciprocal functions are illustrated later.



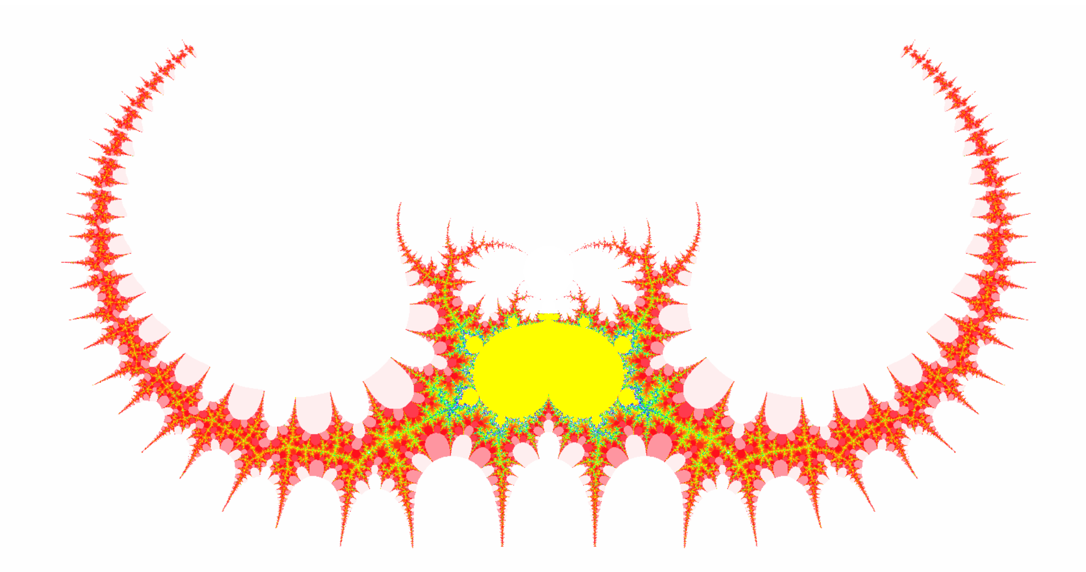
The Exponential Function

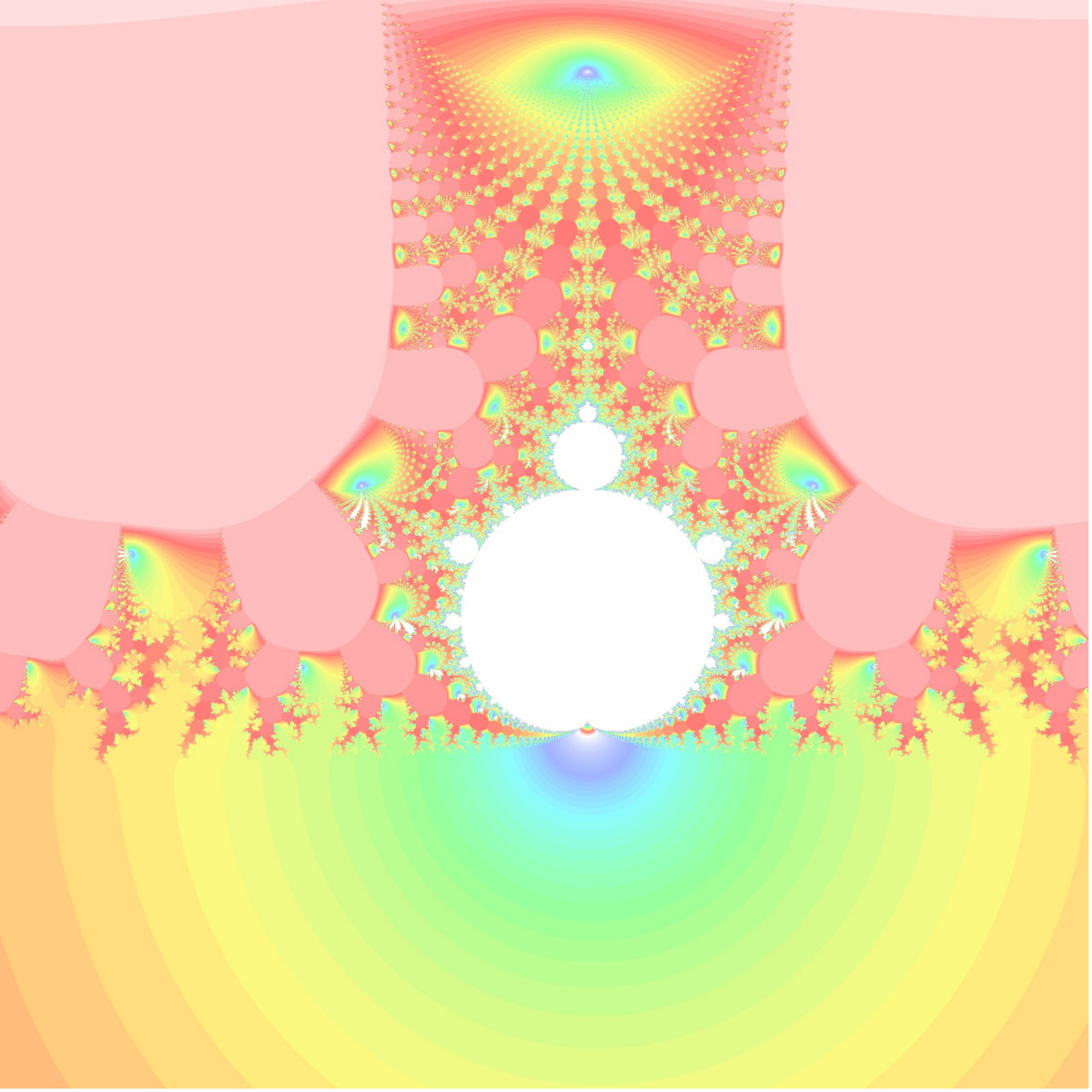
The remarkable image opposite is obtained by iterating the following function:

$$z' = e^{z} - Rz + C$$

with $R = -1.1$. It has been rotated anticlockwise by 90 degrees so the real axis is vertical but, surprisingly, it is not down the axis of symmetry.

The image below is generated by the inverse of the above function, this time with $R = 1$



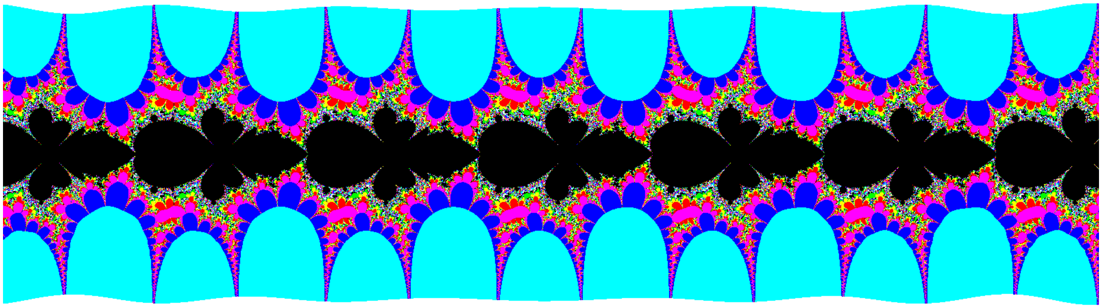


The Cosine Function

Like its real counterpart, the complex cosine function repeats with a period of 2π. Not surprisingly, the function

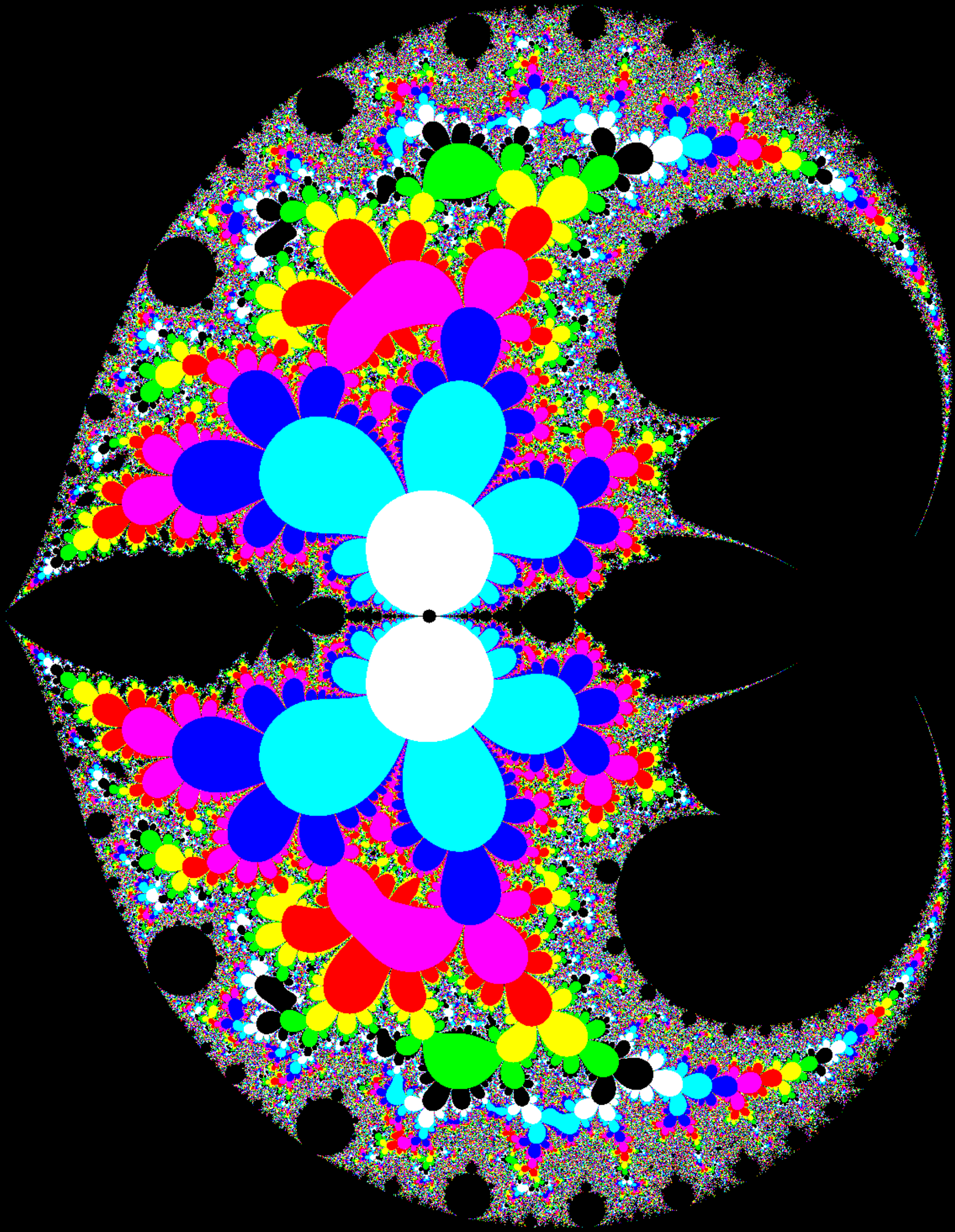
$$z' = \cos(z) + C$$

generates a repeating fractal with a period of 2π as shown below:



When this image is inverted, all of the repeated motifs are compressed into a single dot in the middle while the four prominent spikes which extend into the black body are extended into claws and antennae.

The sine function generates an identical fractal to the one above but shifted by π of course. Also, both functions have two critical points. Using the other one, the repeated motifs point in the opposite direction.



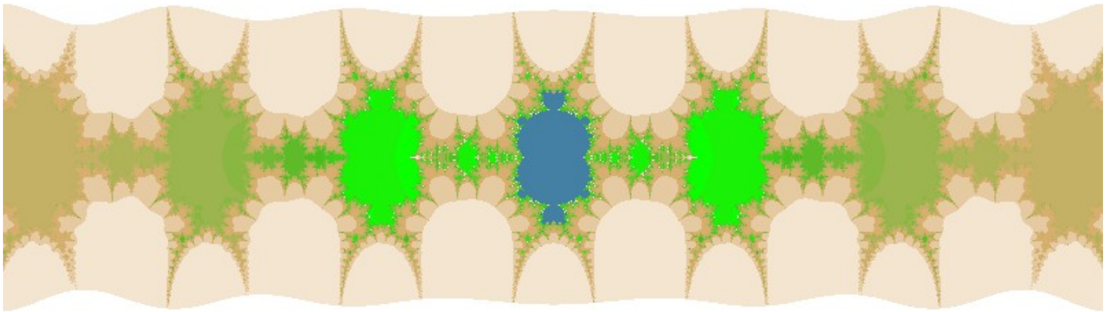
Sine and Tan Functions

Some interesting results can be obtained using the function

$$z' = z - \sin(z) + C$$

(As before, using the cosine only shifts the pattern by π.)

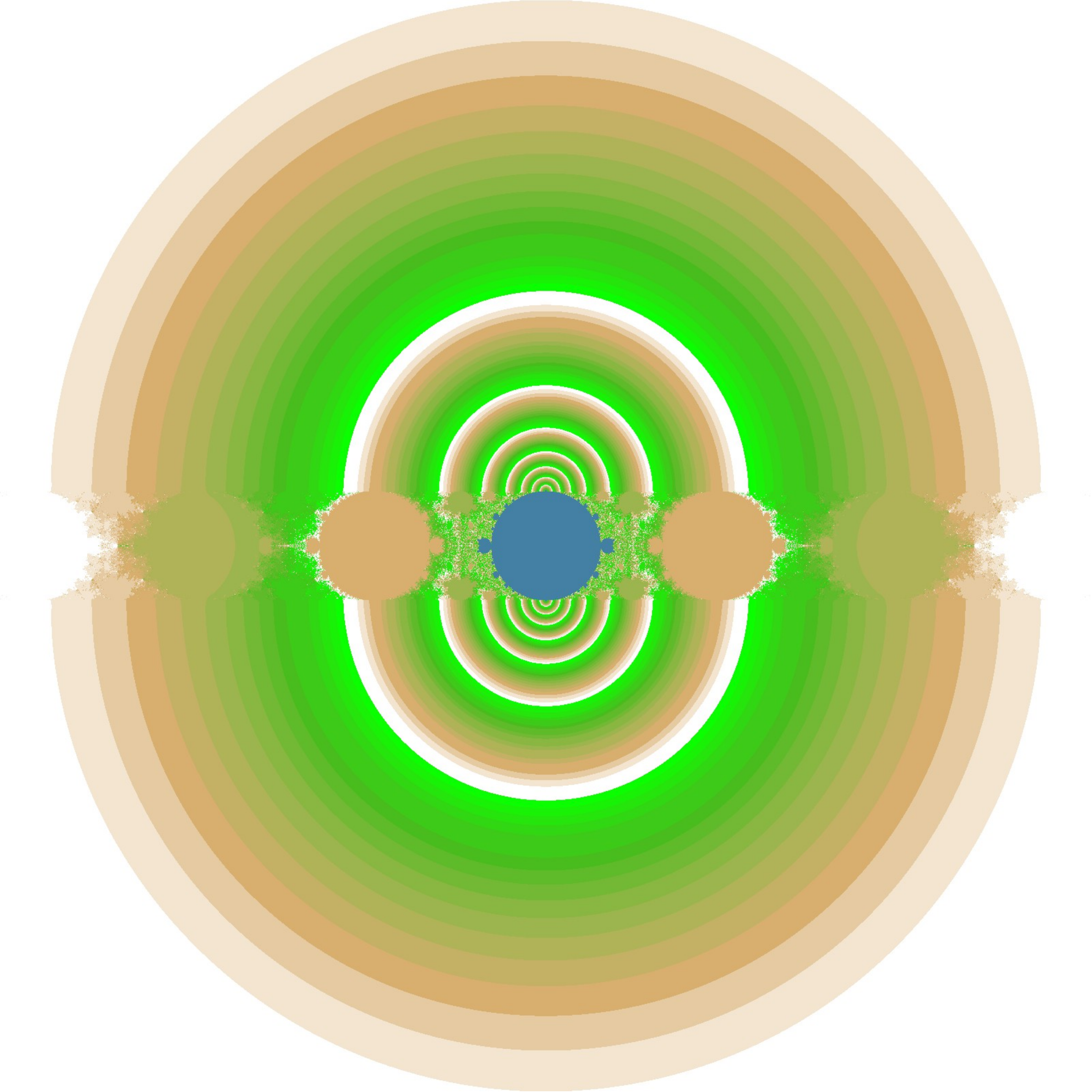
The z term dominates when z is far from the origin so the function is only stable in the central lake around the origin as shown below.



The function

$$z' = z - \tan(z) + C$$

throws up a bit of a surprise. Like the equivalent sin function, only the central lake is stable. What is surprising is the apparently chaotic region that lies on either side on the central lake. Normally nearby values of C escape in about the same time; but here, the function displays extreme sensitivity to initial conditions – a characteristic of deterministic chaos.

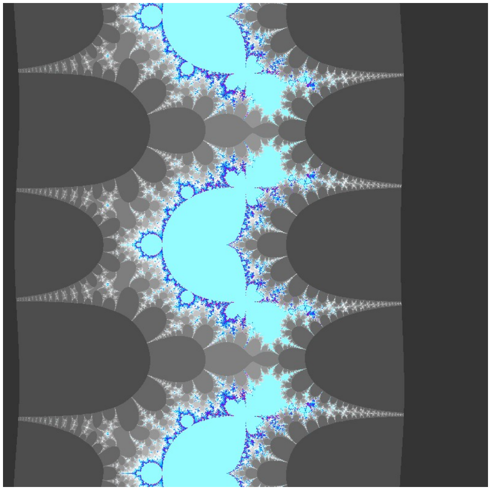


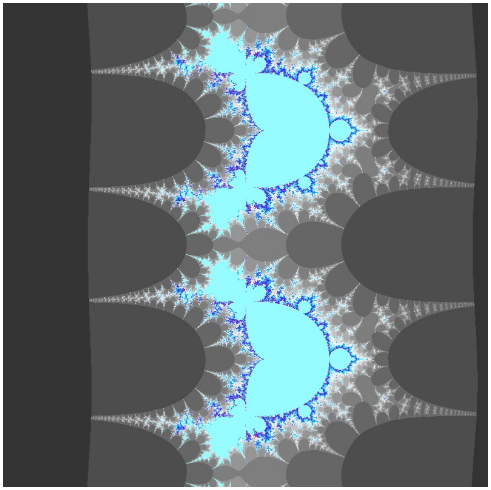
The Hyperbolic Cosine

While the ordinary sin, cos and tan functions repeat along the real axis, the hyperbolic cosine (cosh) repeats along the imaginary axis.

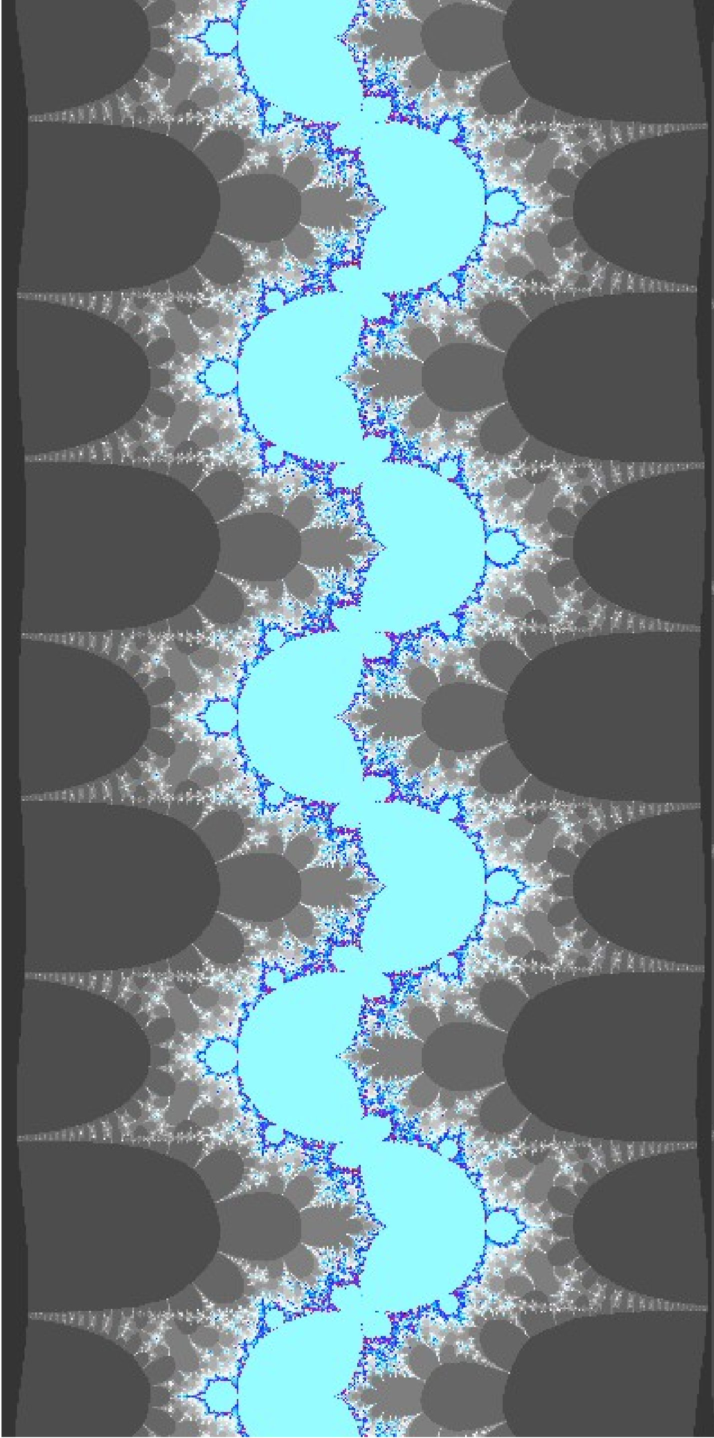
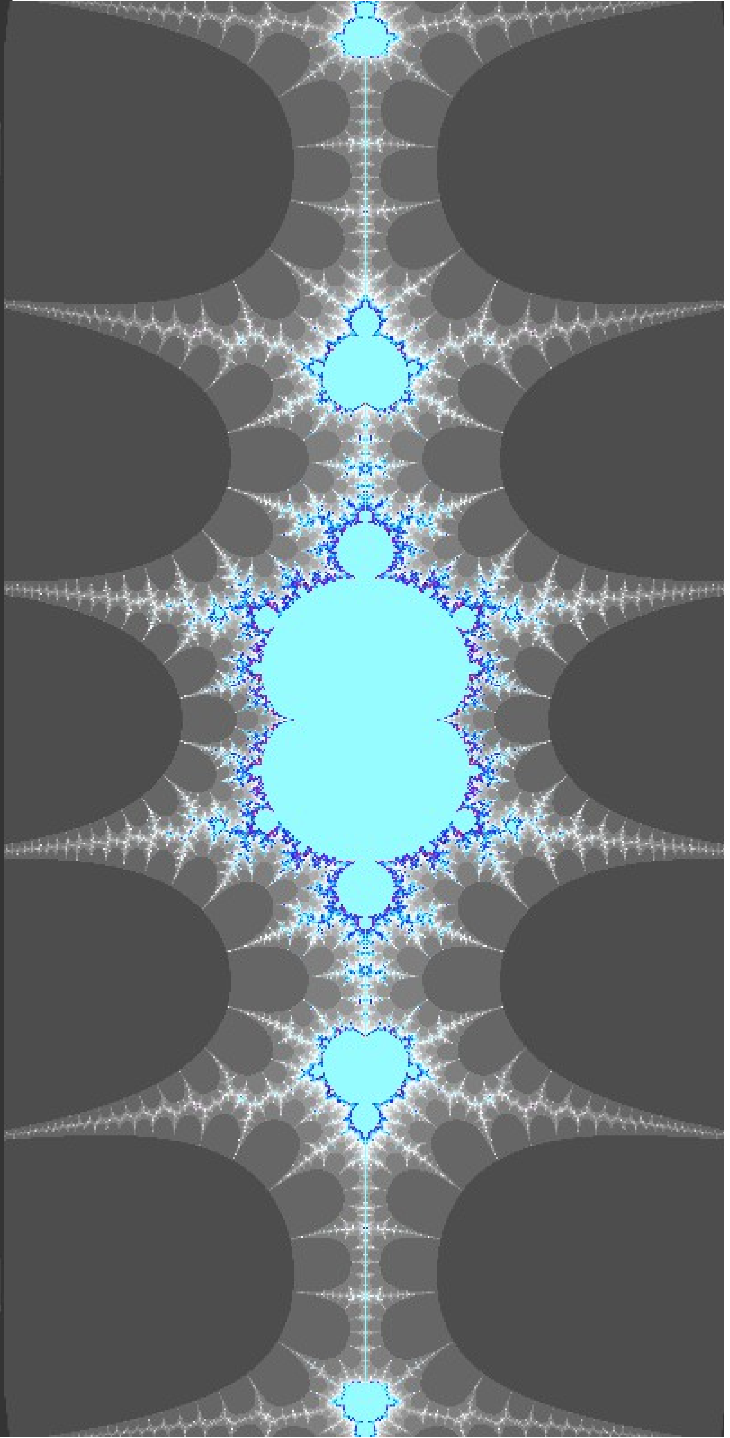
In the two illustrations opposite, the one on the left illustrates the function $z' = \cosh(z) + C$ while the one on the right illustrates $z' = C\cosh(z)$.

The former function has two critical points[1] – i.e. there are two starting starting points which generate a coherent fractal image. These are shown below. The image on the opposite page is a composite of these. Note the incomplete nature of the two 'ears' and the way the two fractals join to form a continuous chain.





1 In fact there are an infinite number of critical points but only two which result in different images.

Squared Mandelbrots

The standard Mandelbrot fractal is a rather odd shape, aesthetically speaking. First it is symmetrical about the horizontal axis whereas, for pretty obvious reasons, humans prefer symmetry about the vertical axis. Now it is easy to rotate the image (as, indeed, I have done several times already) but a more interesting method is to square the parameter C – i.e. use a formula like the following:

$$z' = z^2 + C^2$$

This produces the image at the top right on the opposite page. It may not be terribly beautiful but it is interesting in that it now has rotational symmetry.

Even more fascinating is what happens if we iterate the function

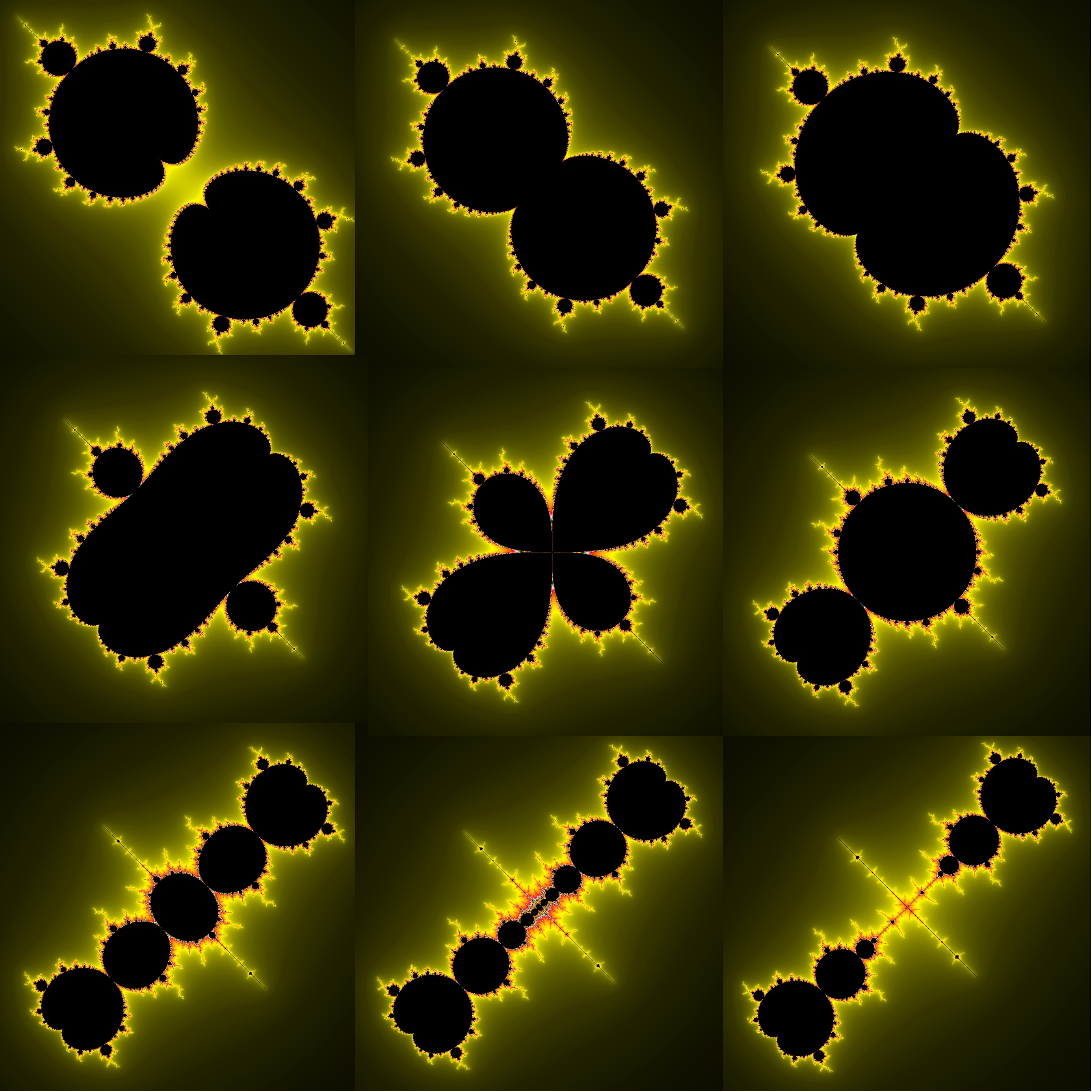
$$z' = z^2 + C^2 + S$$

where S is a complex number. When S is a real number the fractal undergoes a remarkable series of changes. Reading from left to right and top to bottom the images opposite were created with the following values of S:

0.3, 0.2, 0, –0.5, –0.75, –1, –1.3, –1.4 and –1.5

When S is positive the fractal splits into two complete brots which move further and further apart with their antennae pointing away from each other.

When S is negative, the fractal eventually pulls apart into two separate brots facing each other becoming completely separate only when $S < –2$. An interesting transition occurs when R = -0.75



More Squared Fractals

In the top row opposite we explore the effect of imaginary values of S is the equation

$$z' = z^2 + C^2 + S$$

The first image has $S = 0.62i$. At $S = 0.81i$ the fractal is tearing itself into two, a process which is complete when $S = i$.

In the second row we explore the equation

$$z' = z^3 + C^2 + S$$

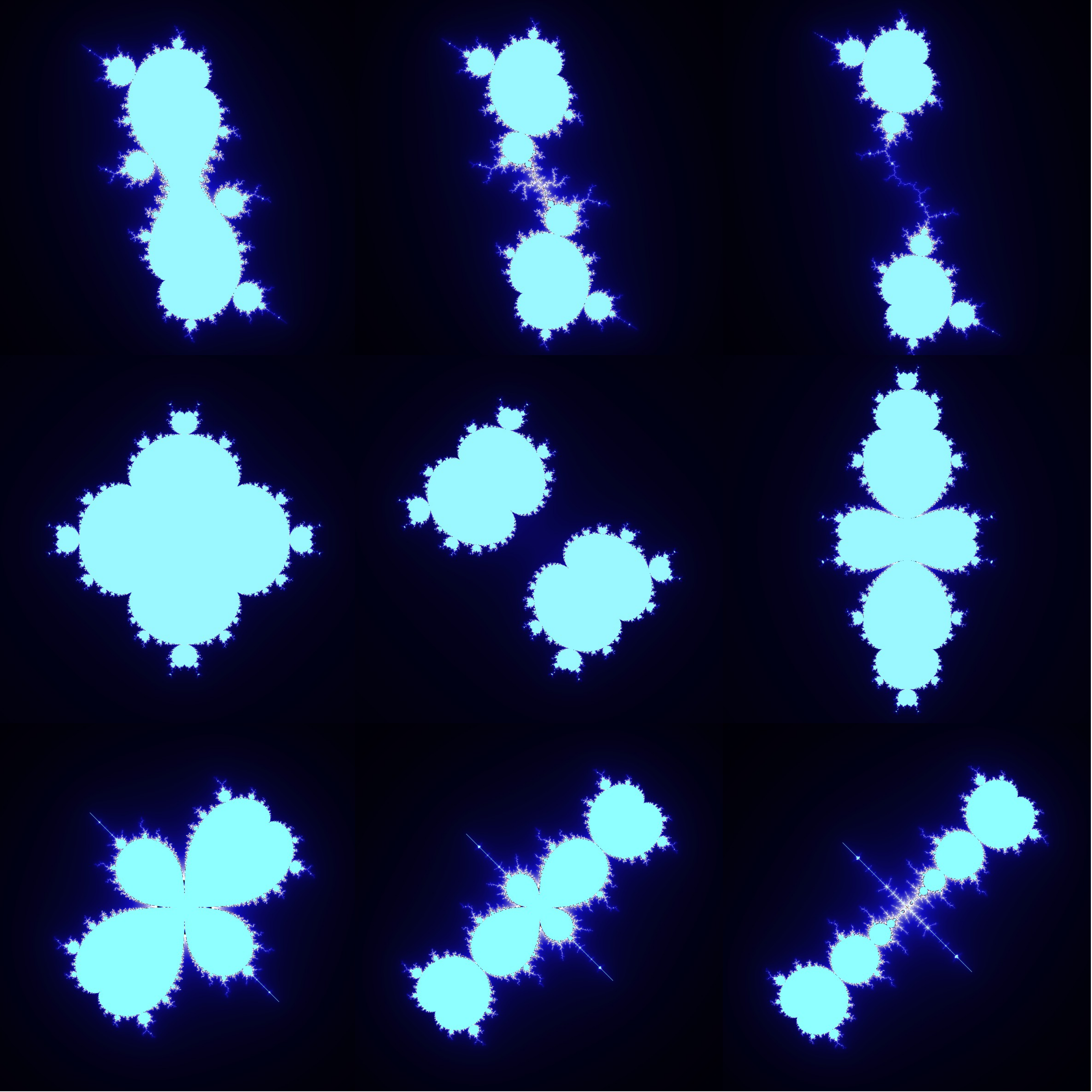
The values of S shown are 0, 0.5 and $0.8i$.

In the bottom row the function used is

$$z' = z^2 + z + C^2 + S$$

with values of S equal to 0, –0.5 and –0.67 respectively.

It is truly amazing how many variations there are of the basic Mandelbrot shape.



Periodicity

Every lobe in a polynomial Mandelbrot fractal has a periodicity – i.e. the point *z* homes in on a periodic cycle.. The main cardioid has a periodicity of one meaning that, when *C* lies in this region, *z* homes in on a single point.

The red lobe to the left of the cardioid has a periodicity of 2 and the point *z* simply zig-zags back and forth.

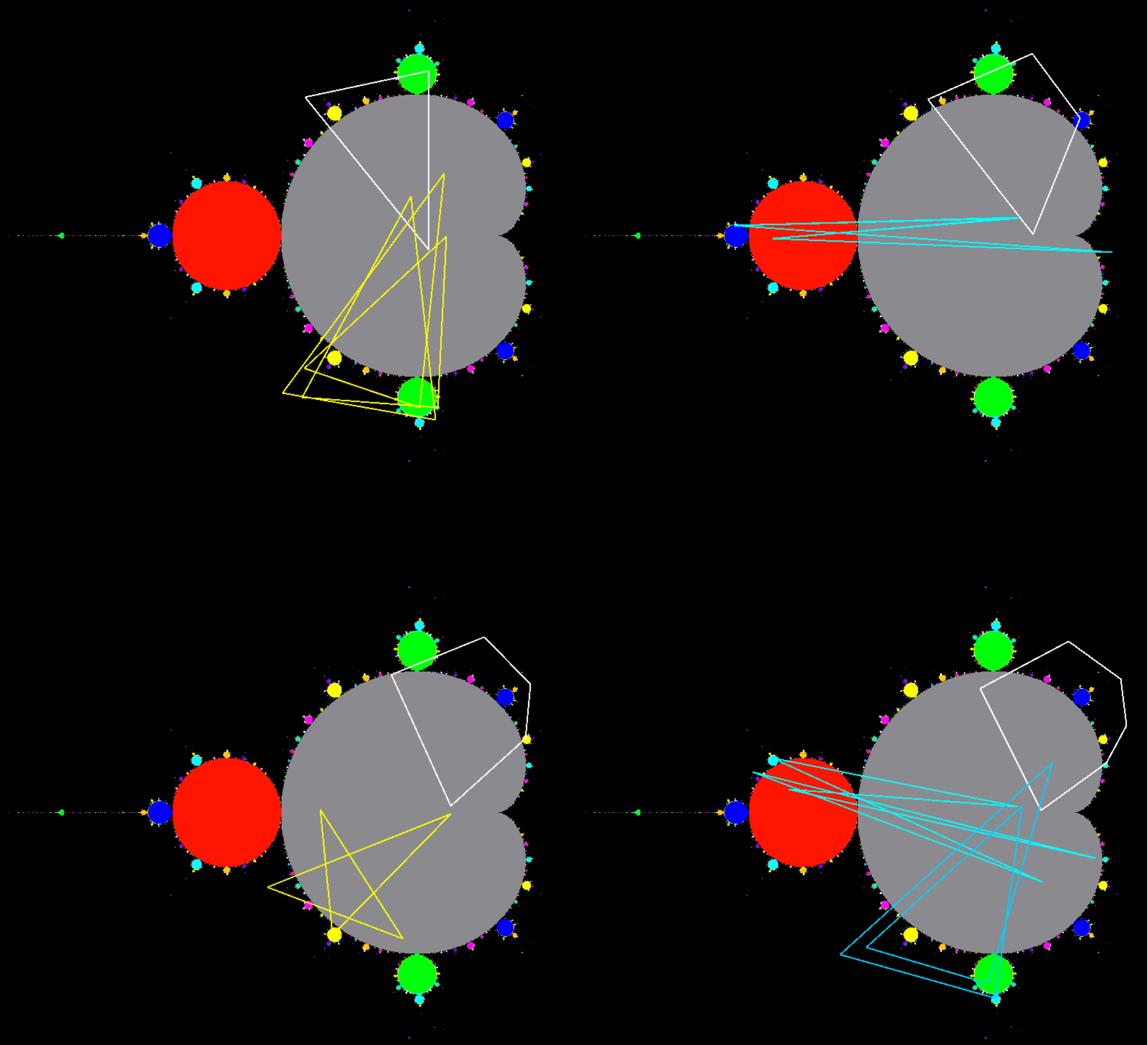
The green lobes above and below the cardioid – and the largest minibrot on the antenna – have a periodicity of 3 (see the first image on the right). When *C* lies near the centre of the lobe, *z* quickly homes in on a triangle as shown in white. If *C* is near the edge of a lobe, the triangle splits. In this case, *C* lies very near the secondary lobe which has a period of 9 so the triangle has split into three.

The blue lobes have a periodicity of 4. All the main sequence lobes on the edge of the cardioid describe simple convex polygons but the blue lobe on the antenna describes an hourglass shape because it is a period 2 split into two.

The yellow lobes have period 5. As this is a prime number it is the first lobe which exhibits a star shape as well as a pentagon.

The last image shows the orbits of lobes with period 6. In addition to the convex hexagon there is a triangle split into two and a zig-zag split into three.

It is a wonderful and little known fact that every lobe has a polygon associated with it and every topologically different polygon has its own lobe!



Periodicity (part II)

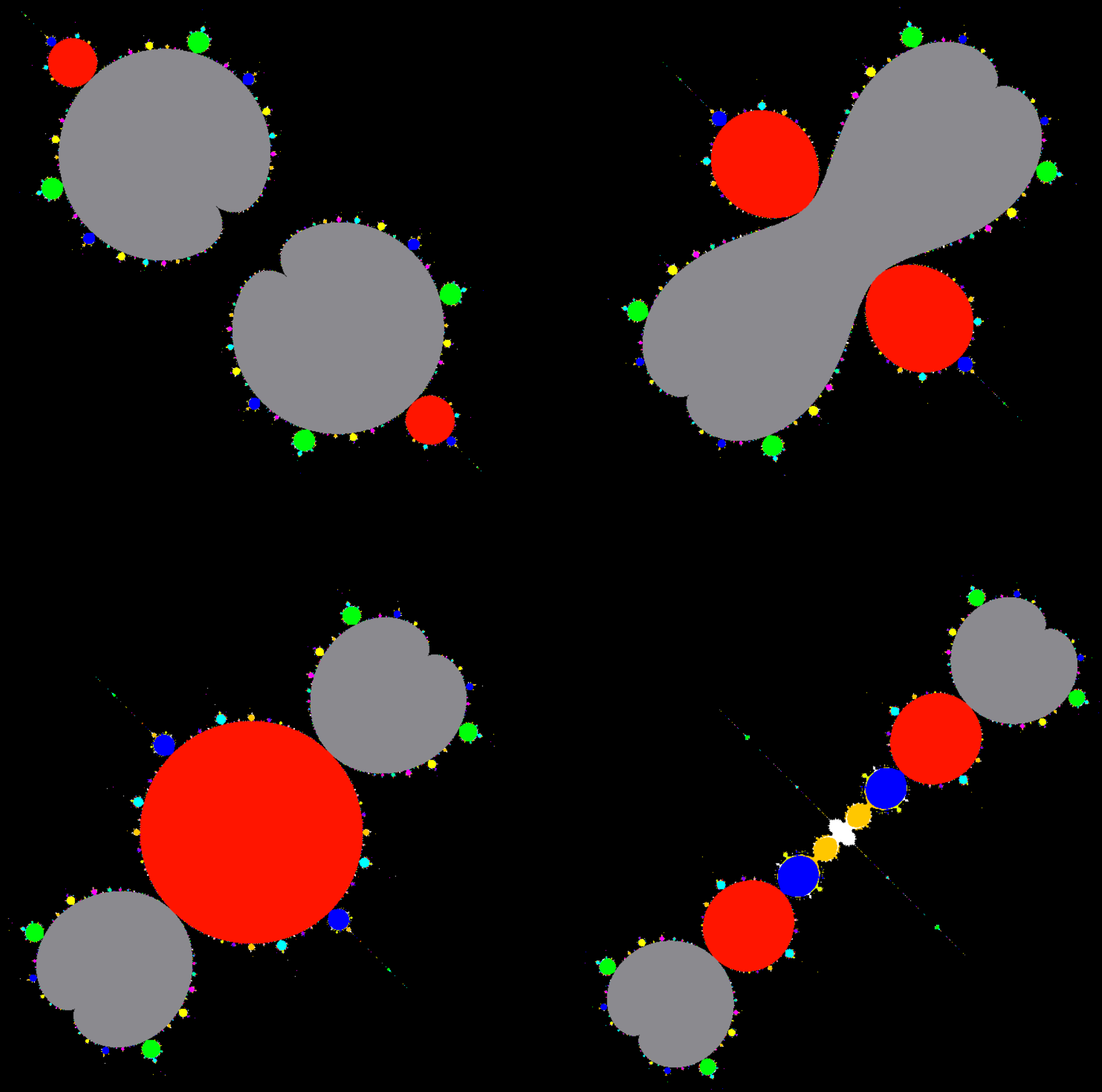
On page 26 we explored the function

$$z' = z^2 + C^2 + S$$

with different real values of *S*. It is interesting to see how, when the squared Mandelbrot function is torn apart using positive values of *S*, the two parts have essentially the same shape and periodicity as the standard Mandelbrot shape. (In the first image opposite *S* has the value 0.3.)

But when *S* is negative, at the pinch point, the central lobe created has a periodicity of 2 – and as the two halves are pulled further and further apart, new central lobes are created with periods of 4, 8, 16 etc. This is why the process when *S* is negative is quite different from the process when *S* is positive.

The subsequent images show the cases when $S = -0.7$, -1 and -1.395



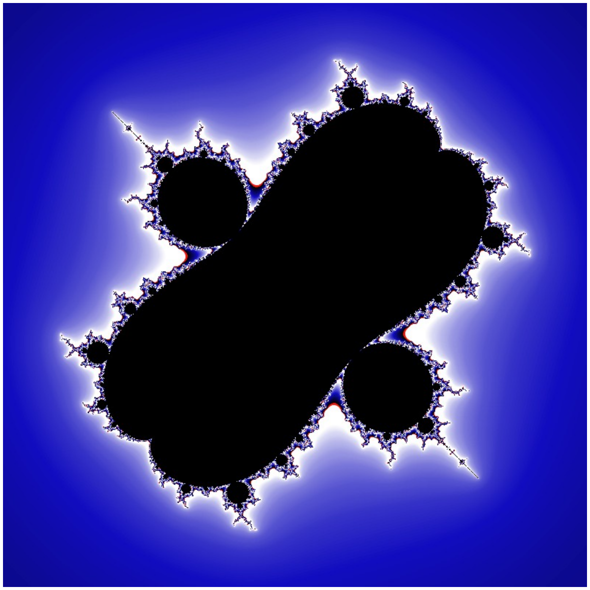
The Double Claw

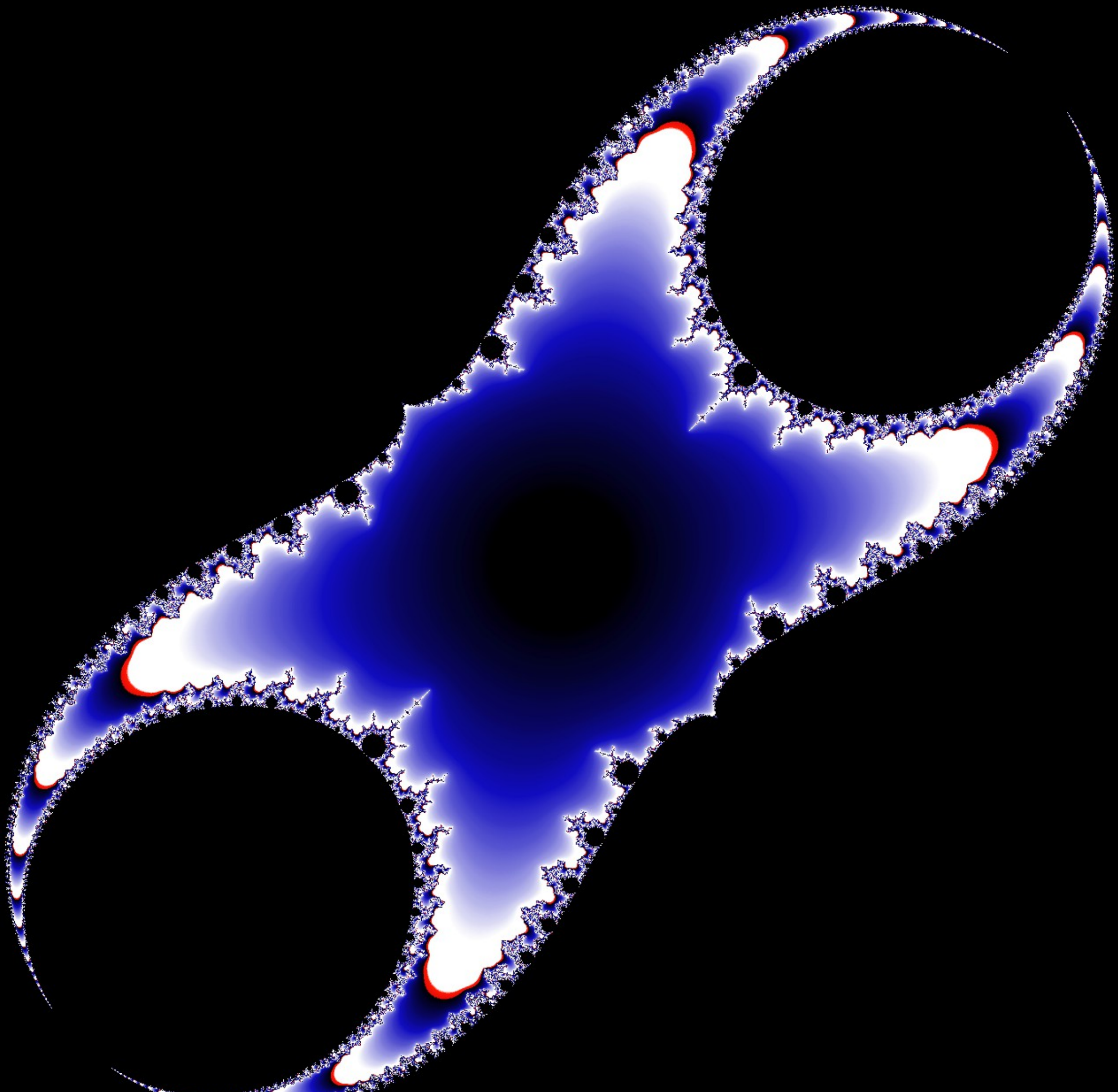
As a final example of a polynomial fractal I present the double claw. It function is:

$$z' = z^2 + \frac{1}{C^2} - S$$

where $S = 0.6$.

It is the inverse of one of the sequence of fractals shown on page 23 – namely:





The Collatz Fractal

In 1937 Lothar Collatz put forward the following conjecture which has no known counterexample but which has never, as yet, been proved. His conjecture was that if you start with any positive integer *n* and carry out the following procedure, you will always end up with the sequence 1:4:2:1...

If n is even, halve it; if n is odd, multiply by 3 and add 1.

It is quite easy to derive a formula which will take any real number *x* and generate a new number which obeys the above rule. One such formula is:

$$x' = (7x + 2 - (5x + 2)\cos(\pi x))/4$$

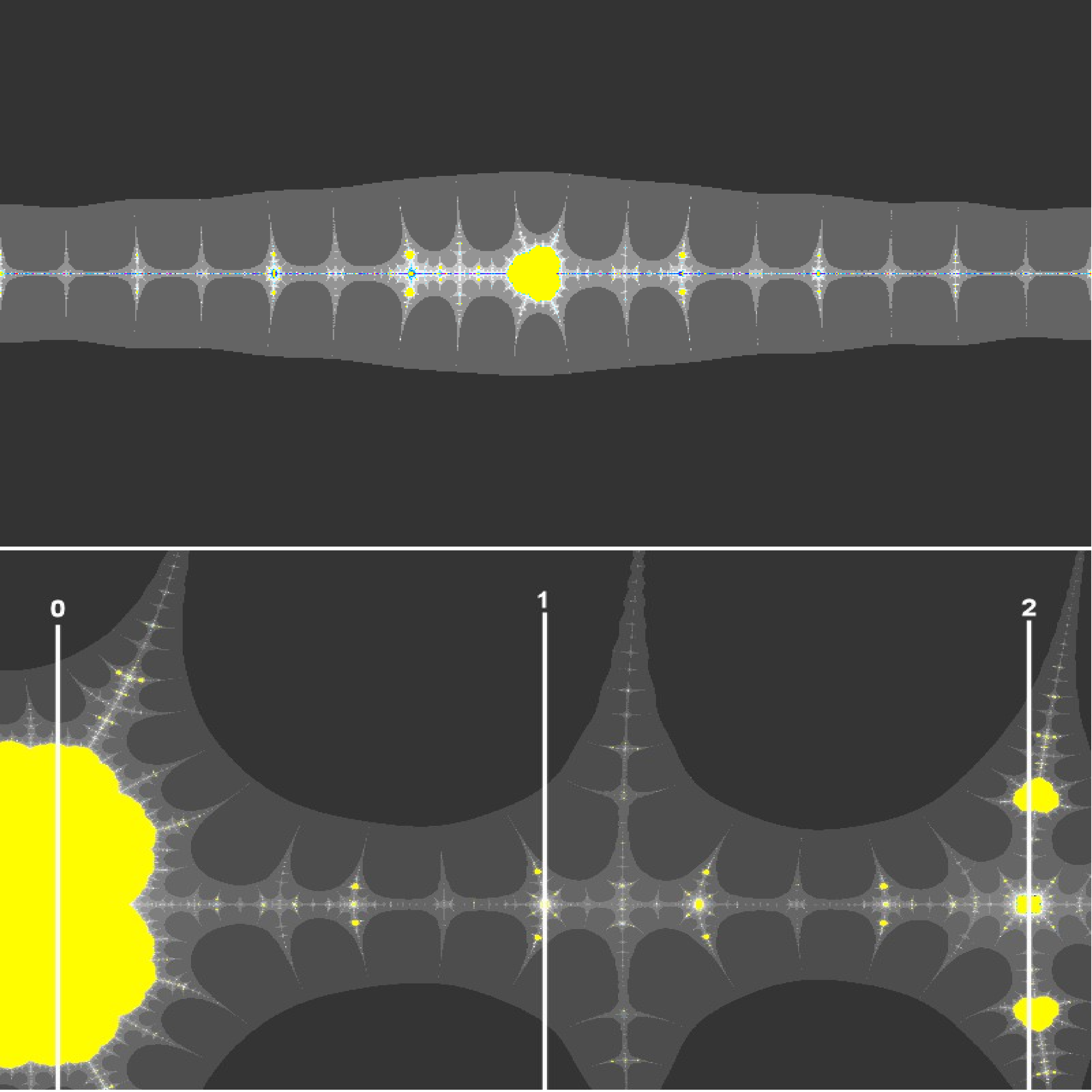
For even numbers $\cos(\pi x) = 1$ in which case $x' = x/2$; but when x is odd, $\cos(\pi x) = -1$ and $x' = (12x + 4)/4 = 3x + 1$.

It is a simple matter to convert this formula into one which can be iterated in the complex plane. I have chosen to use

$$z' = \frac{C}{4}(7z + 2 - (5z + 2)\cos(\pi z))$$

The images opposite are Julia sets – that is to say, they map the behaviour of different starting points for a constant value of C, in this case, 1.

The upper image shows the region from -8 to +8. As expected, there is a stable yellow lake at every integer along the real axis (although, curiously, the integer is not at the centre of the lake, nor is it always on the most prominent line in the region). The lower image shows the region between the origin and +2 in more detail.


0
1
2

Final position

There are lots of ways of colouring the complex plane. The traditional method is to colour the plane according to the number of iterations needed for the initial point to reach a certain distance from the origin – the 'escape' distance. Another way is to colour the plane according to the distance the initial point reaches after a certain fixed number of iterations.

In the images opposite the colour indicates the distance the point z reaches after 3, 4, 5 and 6 iterations using the standard Mandelbrot algorithm

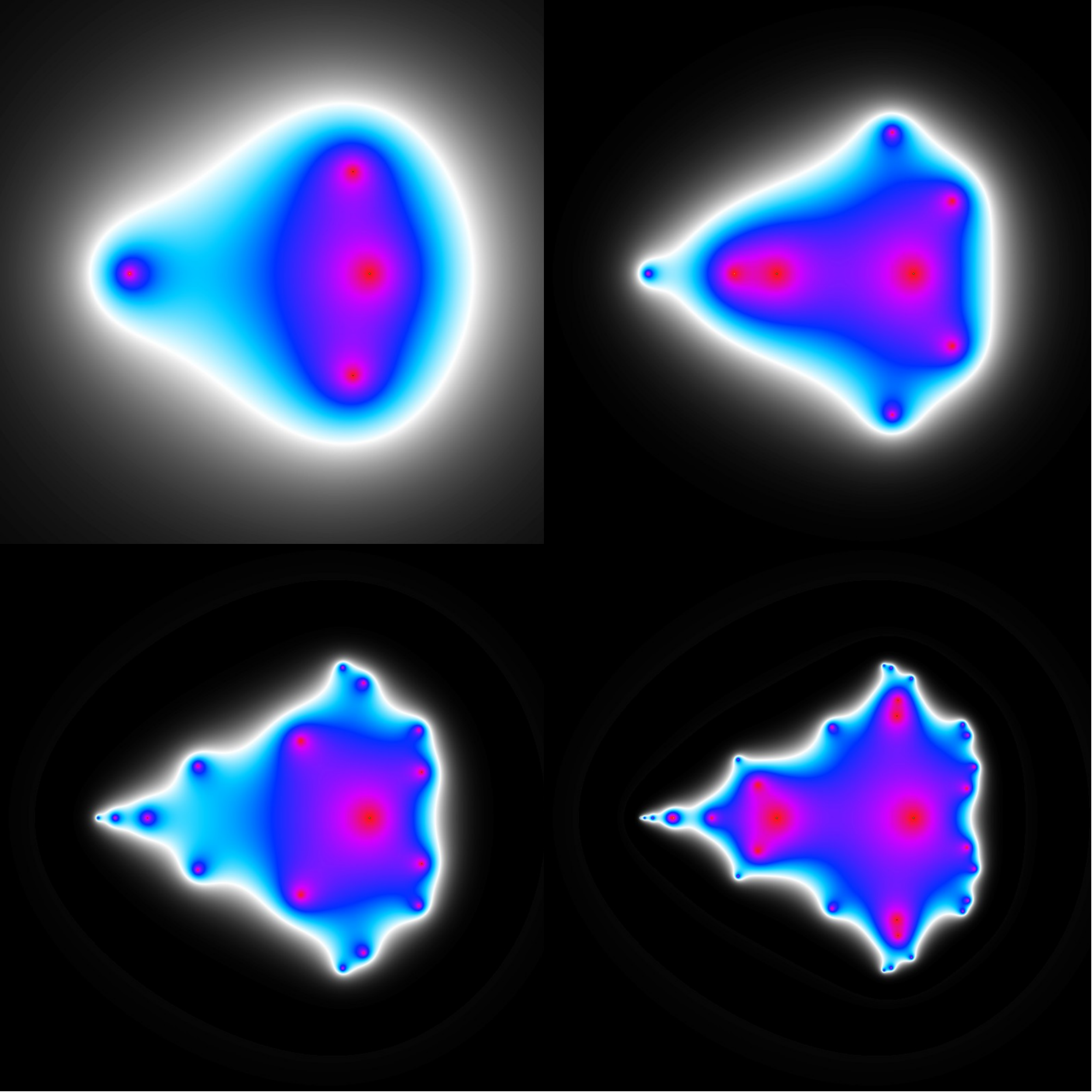
$$z' = z^2 + C$$

Red indicates that the point has returned very close to where it started – i.e. the origin.

There is always a large red spot near the origin because when C is zero, z never moves anywhere. The other red spots reveal the places where z is periodic with a periodicity equal to (or a divisor of) the number of iterations. So the first image picks out the lobes with a periodicity of 1 and 3.

The red blob on the left is due to the small 'brot' along the main 'antenna' which has a periodicity of 3.

This is often a more sensitive method of finding lobes and 'minibrots' with a certain period than trying to measure the period directly.



Final Angle

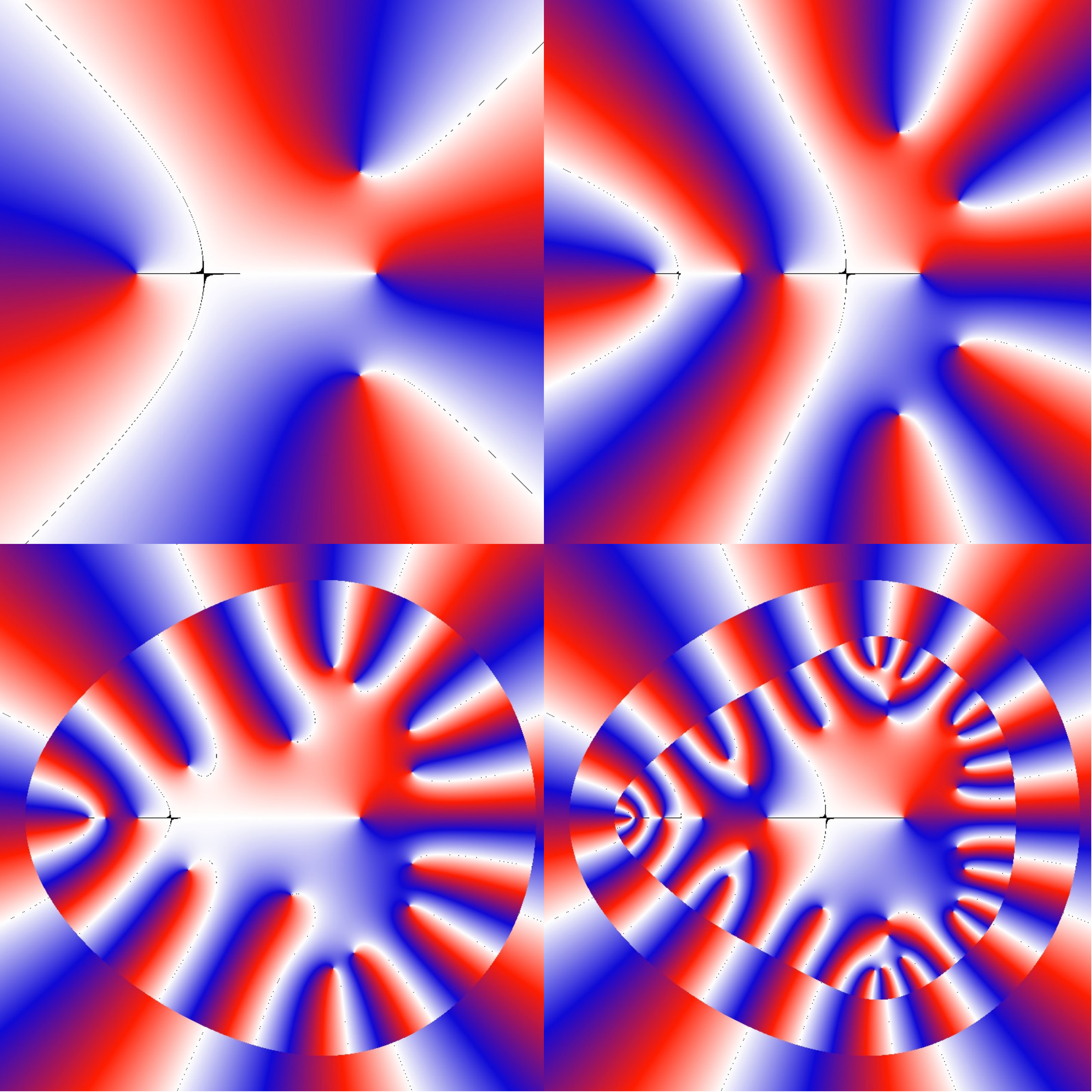
Yet another way to colour code the points is by using the argument of z (i.e. its angular position) after a fixed number of iterations. As before the four images opposite are generated using 3, 4, 5 and 6 iterations respectively.

This time, each lobe or minibrot becomes a centre round which the final angle rotates. In the first image there are just three centres whose periodicity is 3 – the two lobes above and below the main cardioid, the minibrot on the antenna and, of course, the origin which is always included because every number can be divided by 1.

With four iterations there are eight centres with a periodicity of 4. These are most easily counted by counting the number of white 'rays' which reach the edge of the image.

Similarly it is easy to see that there are sixteen centres with a periodicity of 5 and 32 with a periodicity of 6. I have not proved it but it would seem that there are 2^{n-1} centres with a periodicity of n.

(The reason why oval contours have appeared in the images 5 and 6 is that the computer program stops counting iterations when z gets a certain distance away from the origin.. The outer regions are therefore limited to fewer iterations.)



Epsilon spikes

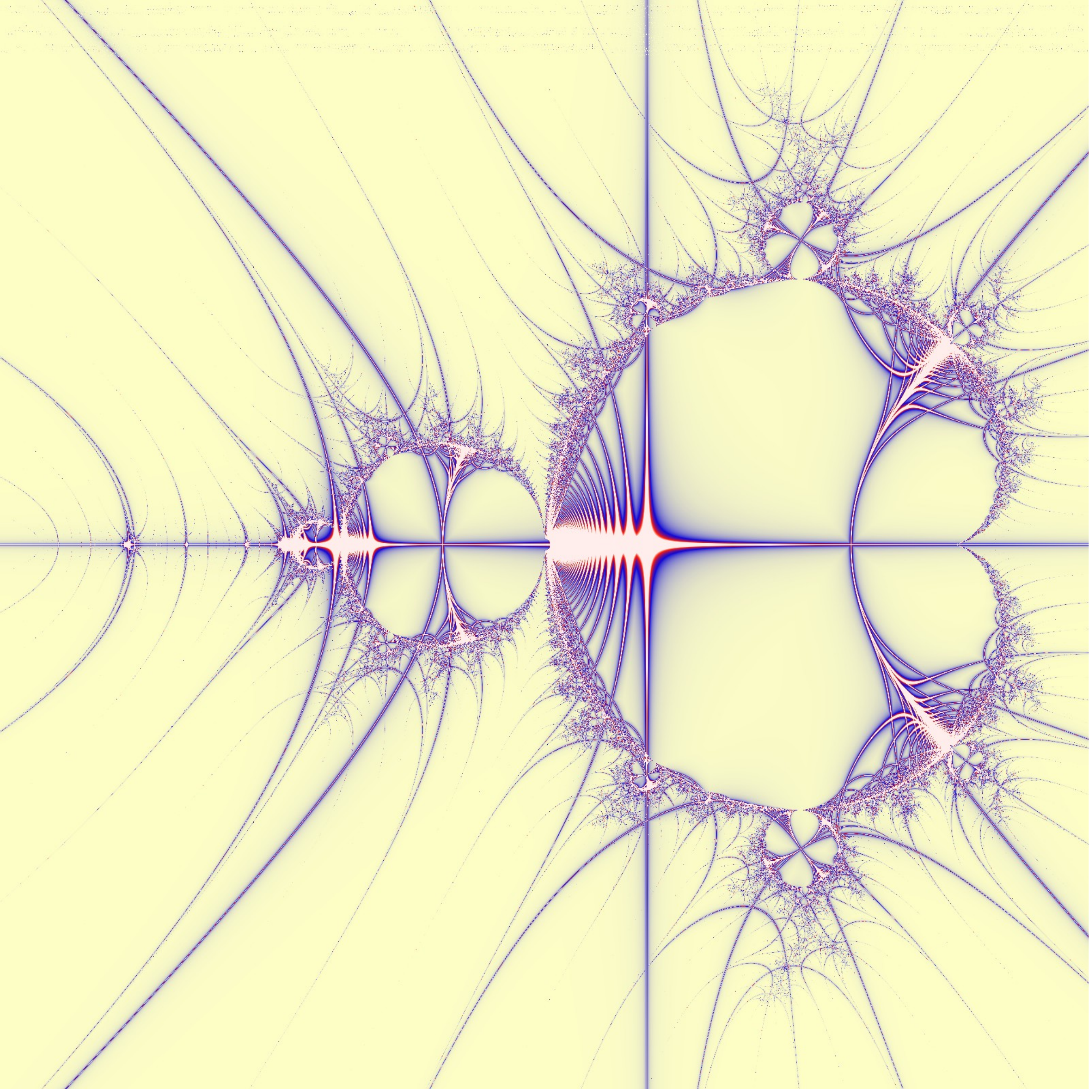
As z wanders over the complex plane, sometimes it comes close to one or other of the axes. This minimum distance called 'epsilon' can be monitored and used to colour the image.

In the image opposite (which uses the standard Mandelbrot algorithm), points which wander close to the axes are picked out in white, red and blue. The values of C which do not cause z to wander close to either of the axes are coloured yellow.

For aesthetic reasons as much as anything else, we have ignored both the initial value of z (which is zero in this case) and also the value of z after its first iteration (which is always C).

The prominent vertical line at the place where $x = -0.5$ is due to the fact that when the real part of C is equal to –0.5, after the second iteration z is always a purely imaginary number – i.e. it always jumps back to the vertical axis. The series of vertical curves to the left of this line is, presumably, due to the fact that values of C here jump back to the vertical axis after 3, 4, 5 etc iterations.

This particular algorithm was invented by Clifford Pickover and images like these are often called Pickover spikes.



More Epsilon Spikes

Top left we have the squared version of the standard Mandelbrot equation – i.e.

$$z' = z^2 + C^2$$

Next to it we have a reciprocal function

$$z' = z + 1/z + C$$

(Note that the origin is at the bottom and the real axis is vertical)

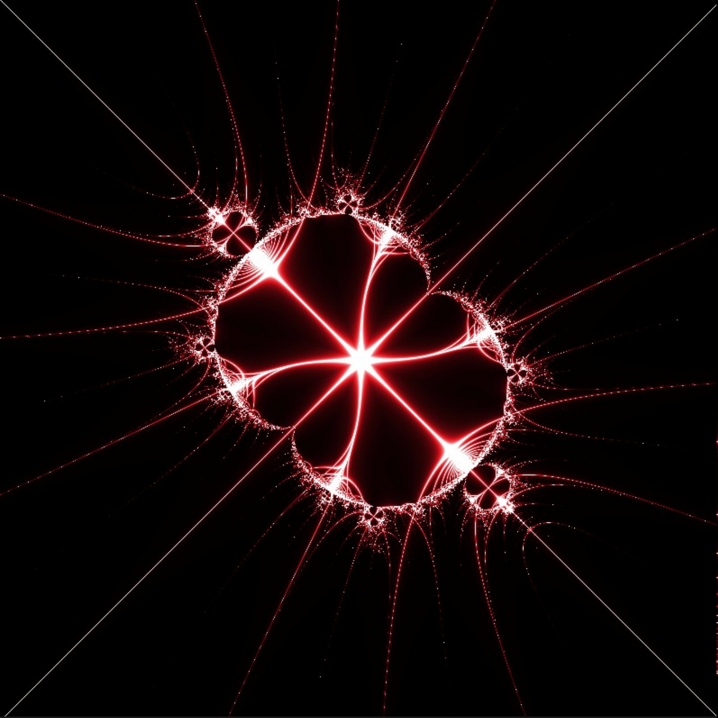
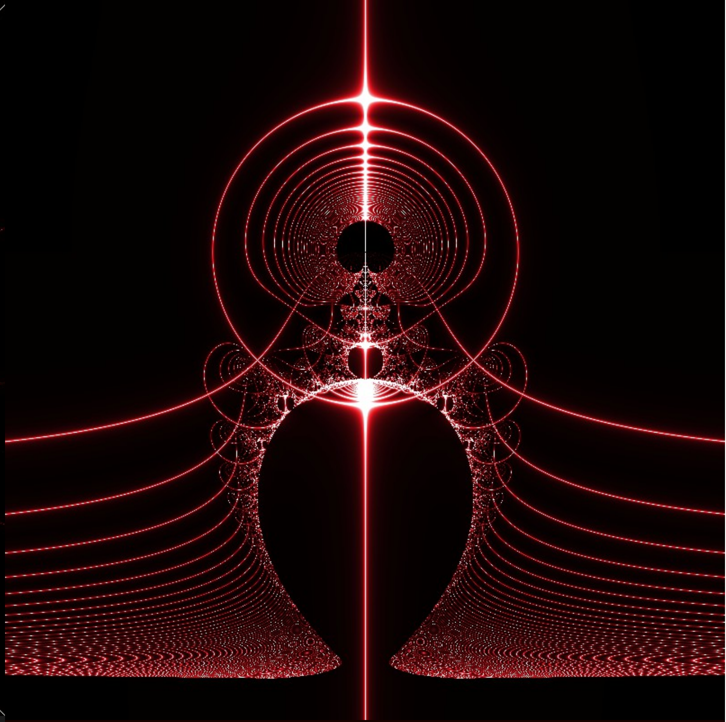
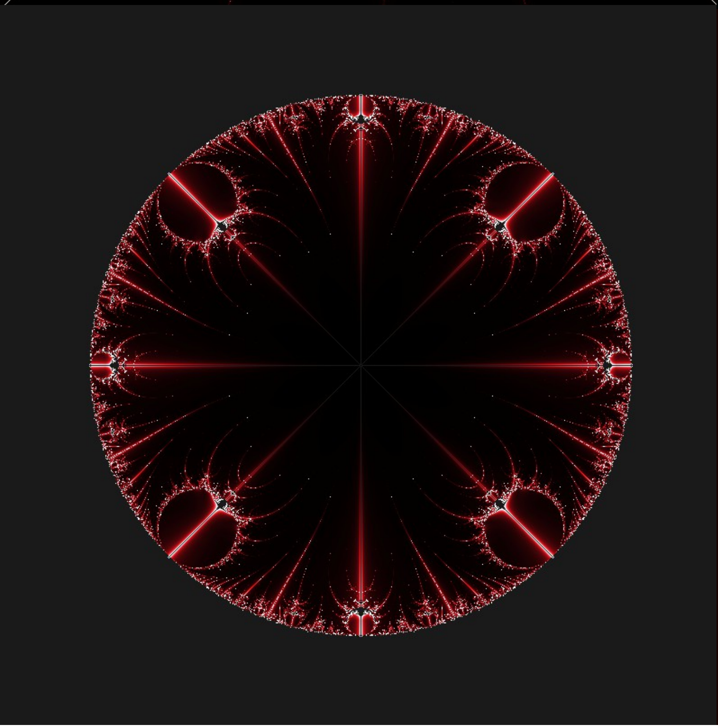
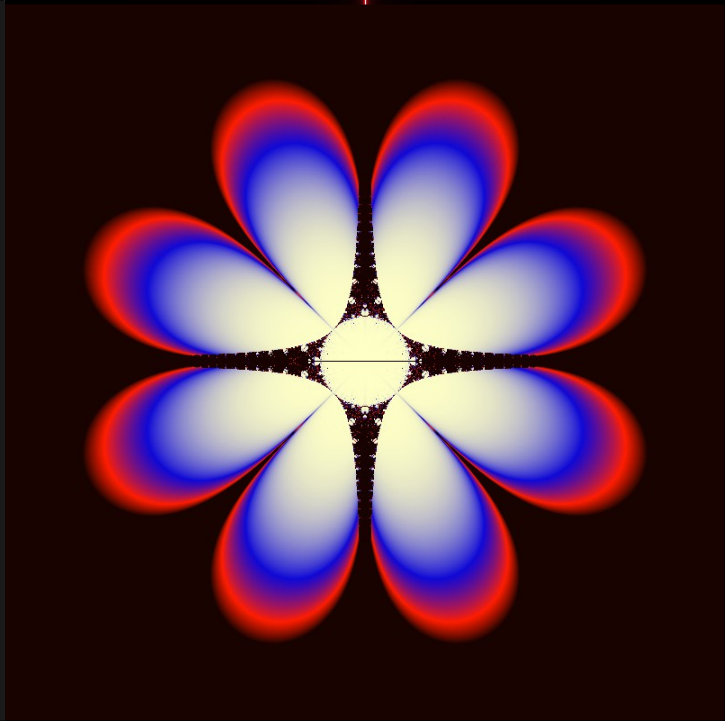
At the bottom we have two inverted and squared functions namely:

$$z' = \frac{1}{C^2}\left(z^3 - z\right)$$

and

$$z' = \frac{1}{C^2}\left(z + \frac{1}{z}\right)$$

Frankly, it is difficult to know where to start with these fractals. They are truly bizarre and quite unpredictable.

Upsilon Dots

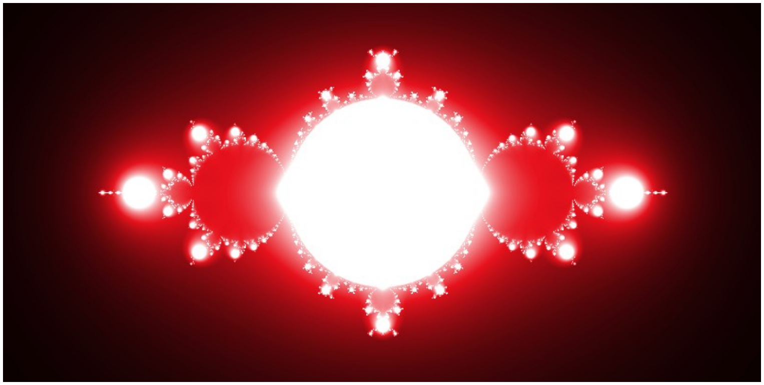
A variation on the idea of Epsilon Spikes is what I call Upsilon Dots. Instead of monitoring how close z approaches either of the axes we simply monitor how close it gets to the origin. (I refer to this distance as upsilon.)

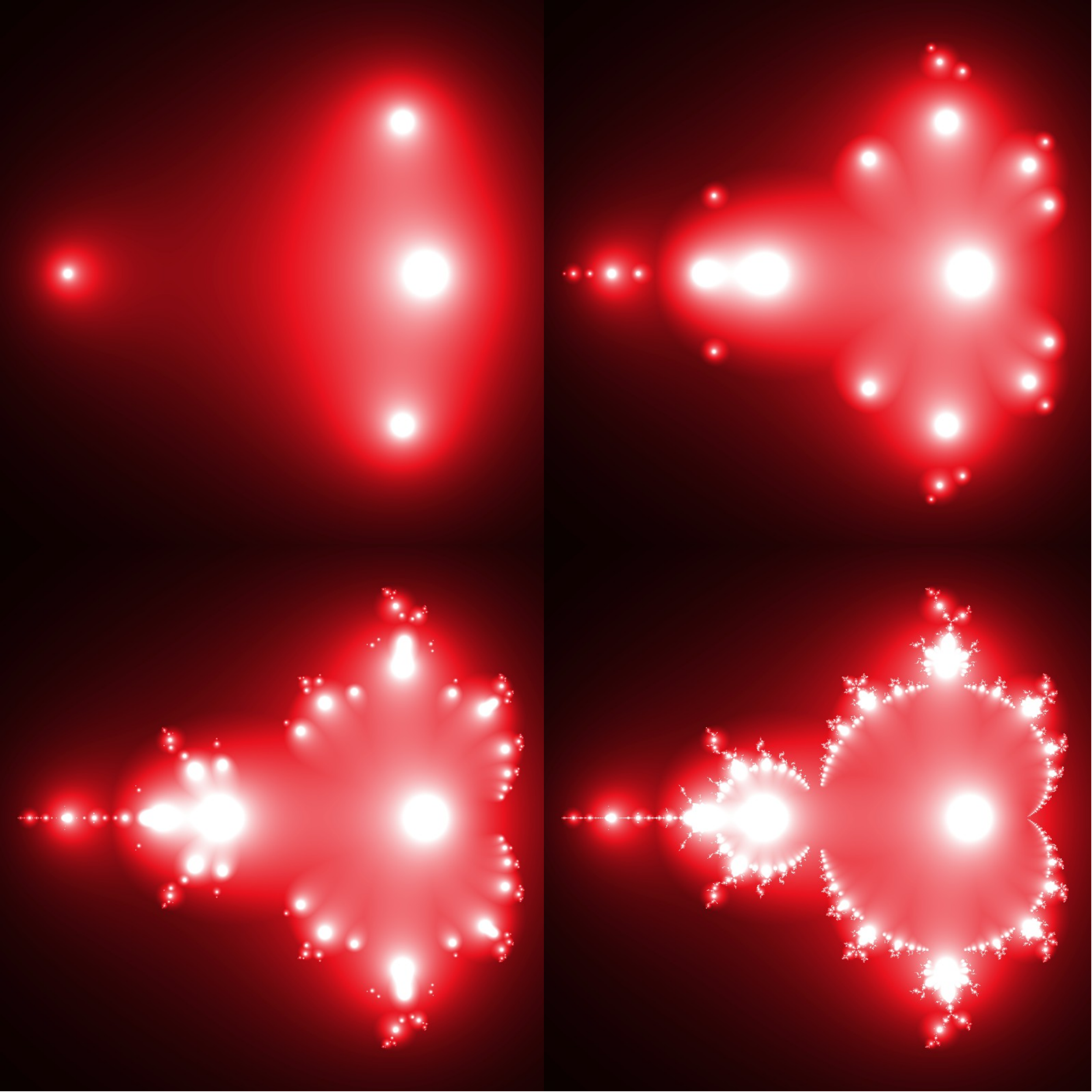
In the images opposite, we explore how close z get to the origin after 3, 5, 8 and 10 iterations.

Now in the case of the standard Mandelbrot algorithm, values of C which are centred inside any of the lobes always return to the origin after precisely p iterations where p is the periodicity of the lobe. If we limit the number of iterations to 3 as in the first image on the right, then only those lobes with a periodicity of 3 (or 1) will stand out.

Limit the number of iterations to 5 and all the lobes with periodicities of 1, 2, 3, 4 and 5 will light up.

The image below shows an Upsilon Dot map of a cubic function.





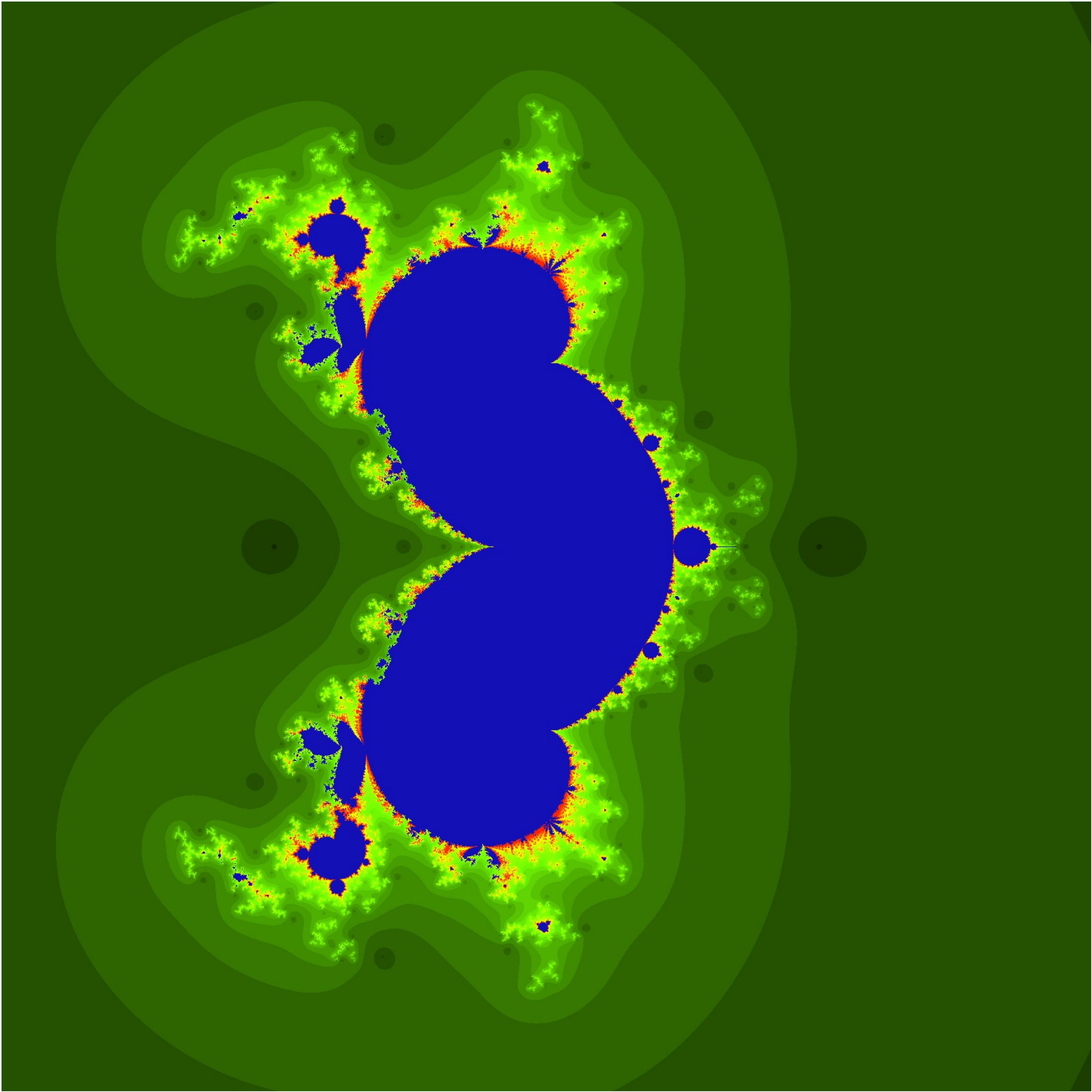
Rational Polynomials

A rational polynomial is a function of the form $f(z) = \frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomial functions of z with real coefficients. Of critical importance is the order of the two functions $p(z)$ and $q(z)$. The classic Mandelbrot set can be considered to be generated by a rational polynomial in which $p(z) = z^2$ and $q(z) = 1$. Here the order of the numerator is 2 and the order of the denominator is zero. In general we can say that the order of a rational polynomial is equal to the order of the numerator minus the order of the denominator. Any rational polynomial whose order is greater than zero will, in general, generate a fractal after the manner of the Mandelbrot set but it will be distorted and may be fragmented as well.

For example, the image opposite is generated by the following algorithm:

$$z' = \frac{z^4 + 2z^2 - 1}{z^2 - 1} + C$$

Since $p(z)$ has order 4 and $q(z)$ has order 2 this function has order 2 so it has a stable region. One of its critical points is the origin and this is the one that has been used here. As discussed on page 8 any polynomial with real coefficients whose critical points are also real will be symmetrical about the real axis.



A Polynomial of order 1

Consider the function

$$z' = z + 1/z + C$$

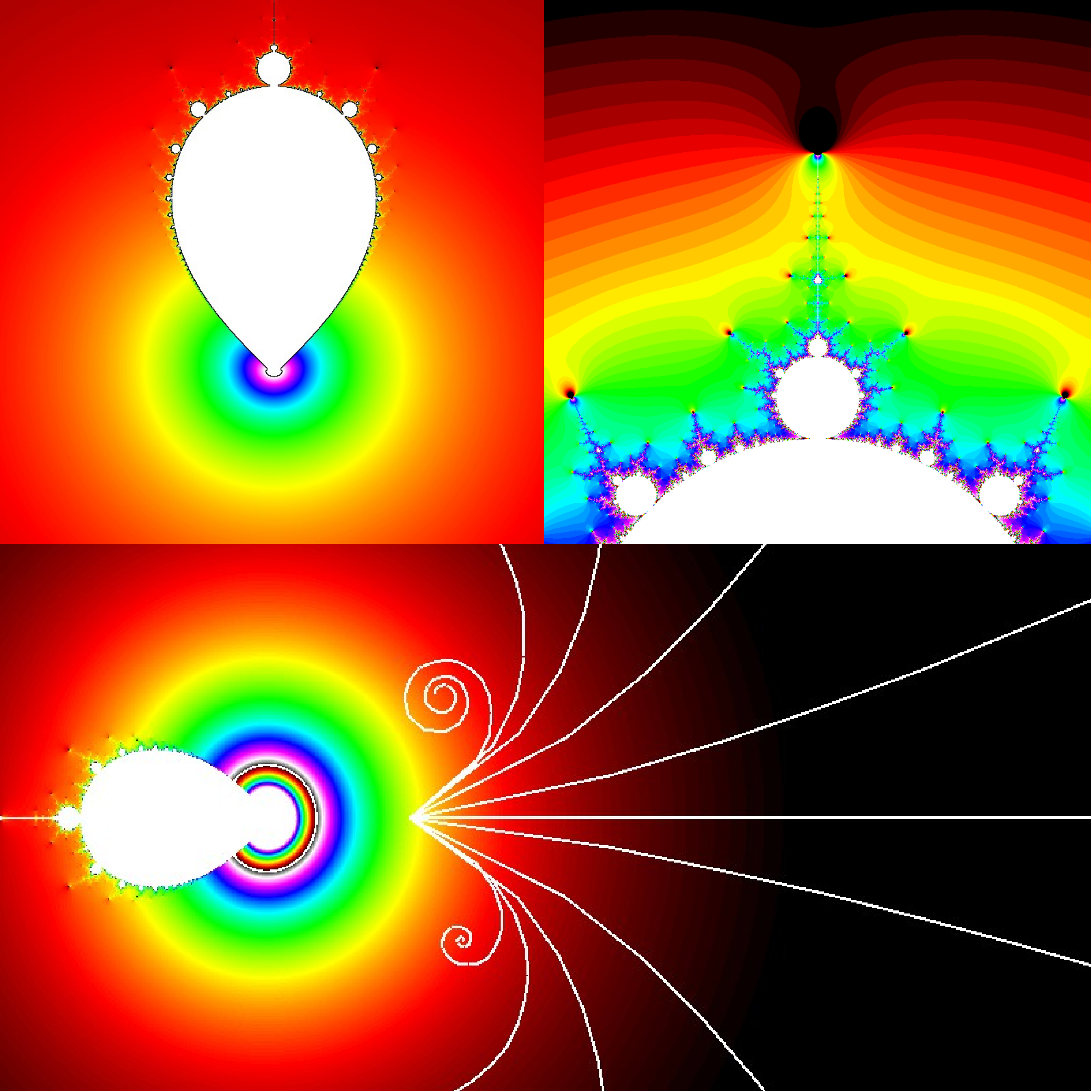
This can equally well be written as

$$z' = \frac{z^2+1}{z} + C$$

It has a critical point at (1, 0) and its stable set looks like a hot air balloon. (The image has been rotated clockwise by 90°. The other image at the top shows a magnified section.)

When $C = 0$, the sequence goes 1 => 2 => 2.5 => 2.9 => etc. which diverges off to infinity incredibly slowly. That is why there is a kind of 'bulls-eye' round the origin. However, when C lies inside the stable area, the path of z curls ound and homes in on the point which is the solution to the equation $z + 1/z + C = z$. It is easy to see that this is just $-1/C$.

The image at the bottom shows the unrotated fractal together with a number of rays showing how the point z iterates using different values of C all taken from the dark blue ring which is an approximate circle of radius 0.5 centred on the origin. All the rays start at the critical point $z = 1$ – i.e. the point on the real axis which is coloured yellow. As C moves round the dark blue ring, the rays become progressively more and more curved, finally homing in on a solution when C enters the stable region.



A Polynomial of order –2

The function $\frac{1}{z^2 + a}$ has order –2. In general, functions of order less than or equal to zero do not generate Mandelbrot fractals using the 'escape' algorithm because the iterative function

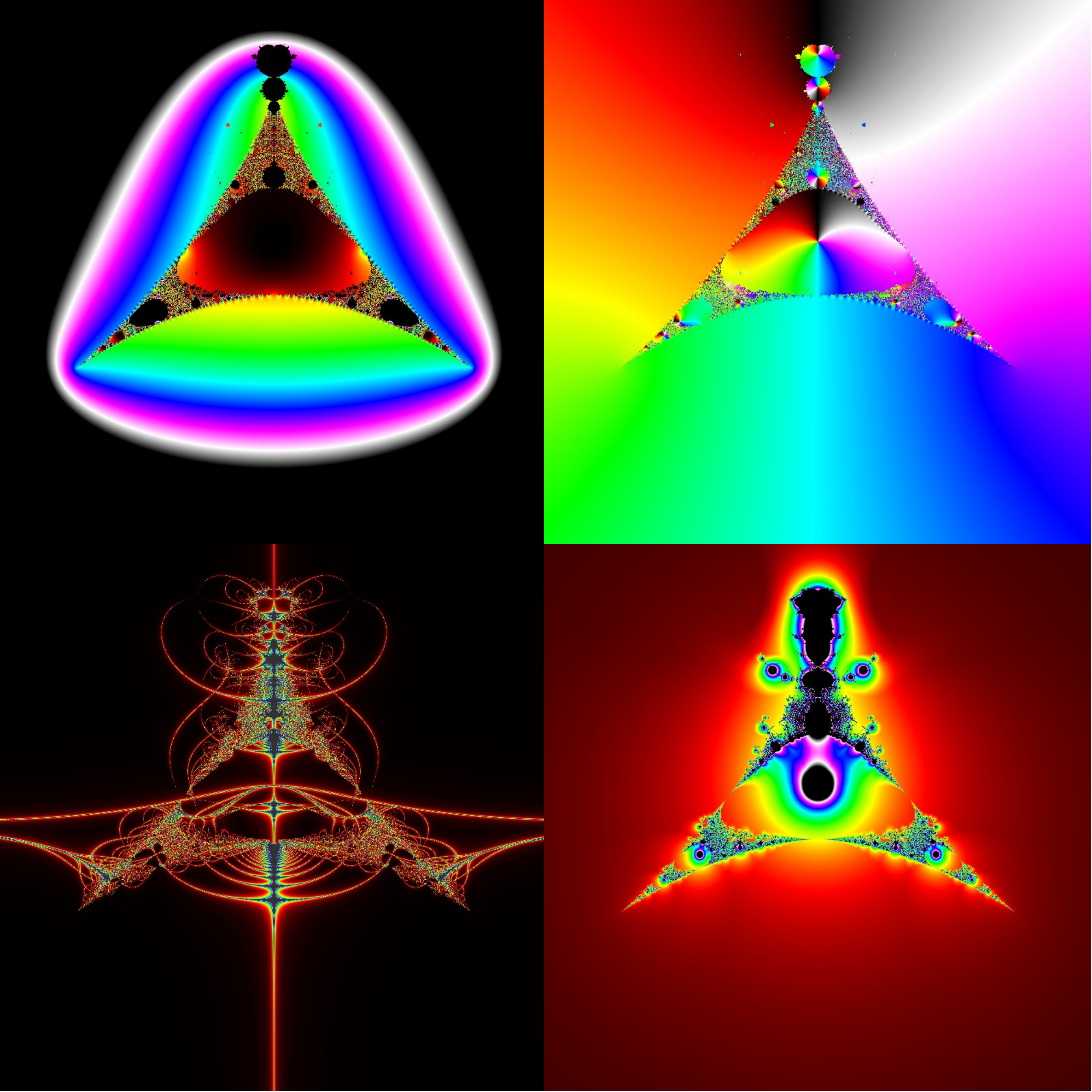
$$z' = \frac{1}{z^2 + a} + C$$

is stable for all values of *C*.

In order to tease out its structure we must employ one of the other methods of turning its output into colour. The four illustrations opposite show the results of using the modulus, argument, epsilon spike and upsilon dot algorithms[2] using a value of $a = 0.5$. (All of them have been rotated clockwise so the real axis is vertical.)

The first (the modulus algorithm) shows how far *z* has wandered away from the origin after 100 iterations. When *C* is close to zero, *z* qauickly enters a periodic orbit of order 2 and the colour is black. However, on the boundary of the bell-shaped curve there is a discontinuity. When *C* lies outside the triangular region, *z* can do some crazy things before settling onto a stable point which is not that far from *C* itself. Inside the triangle there are stable and highly chaotic regions.

2 For an explanation of these algorithm see page 38 and the following pages.



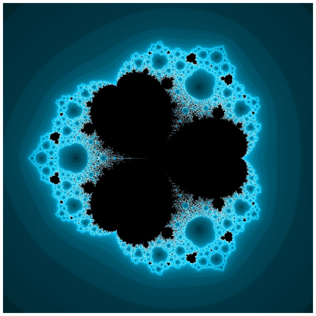
Multiplicative Polynomials

Illustrated below and opposite is the Mandelbrot set for the function

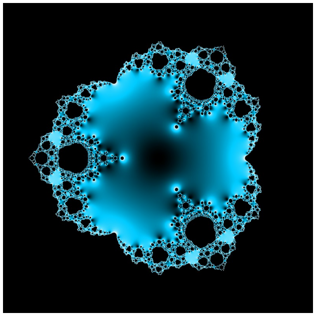
$$z' = C\frac{z^3 + 1}{z}$$

(Note that the function is multiplied by C rather than added. The latter option is relatively uninteresting.)

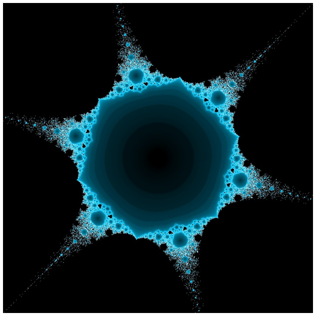
Illustration A is the standard 'escape' algorithm showing the chaotic 'foam' characteristic of a reciprocal function. B shows the escape distance after 8 iterations. C is the 'escape' algorithm with *C* inverted and squared. Opposite is the epsilon spike algorithm using *C* squared



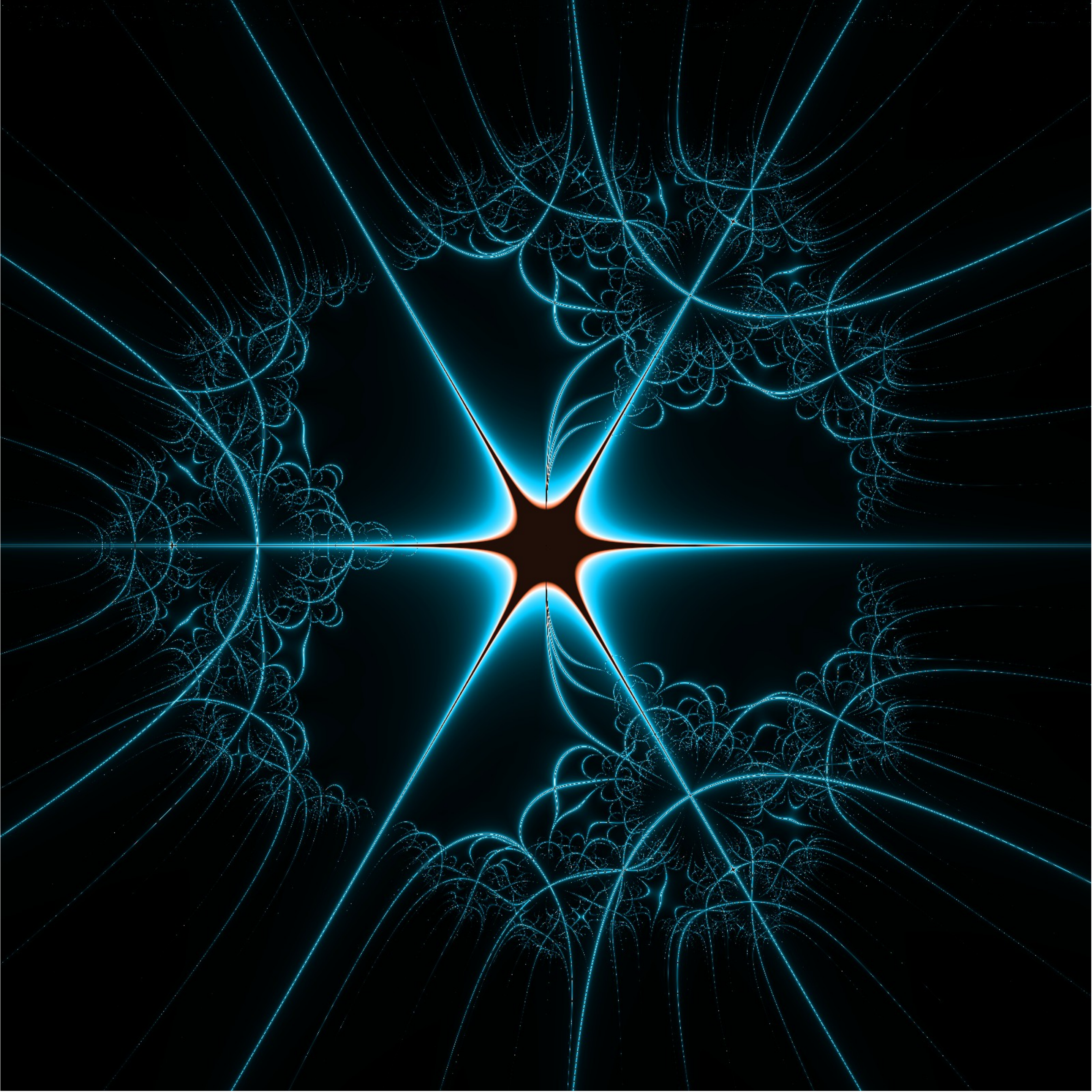
A



B



C



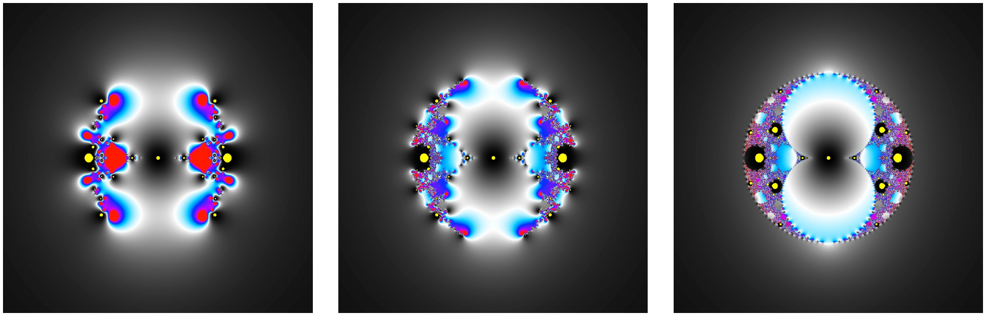
A Multiplicative Polynomial of order -2

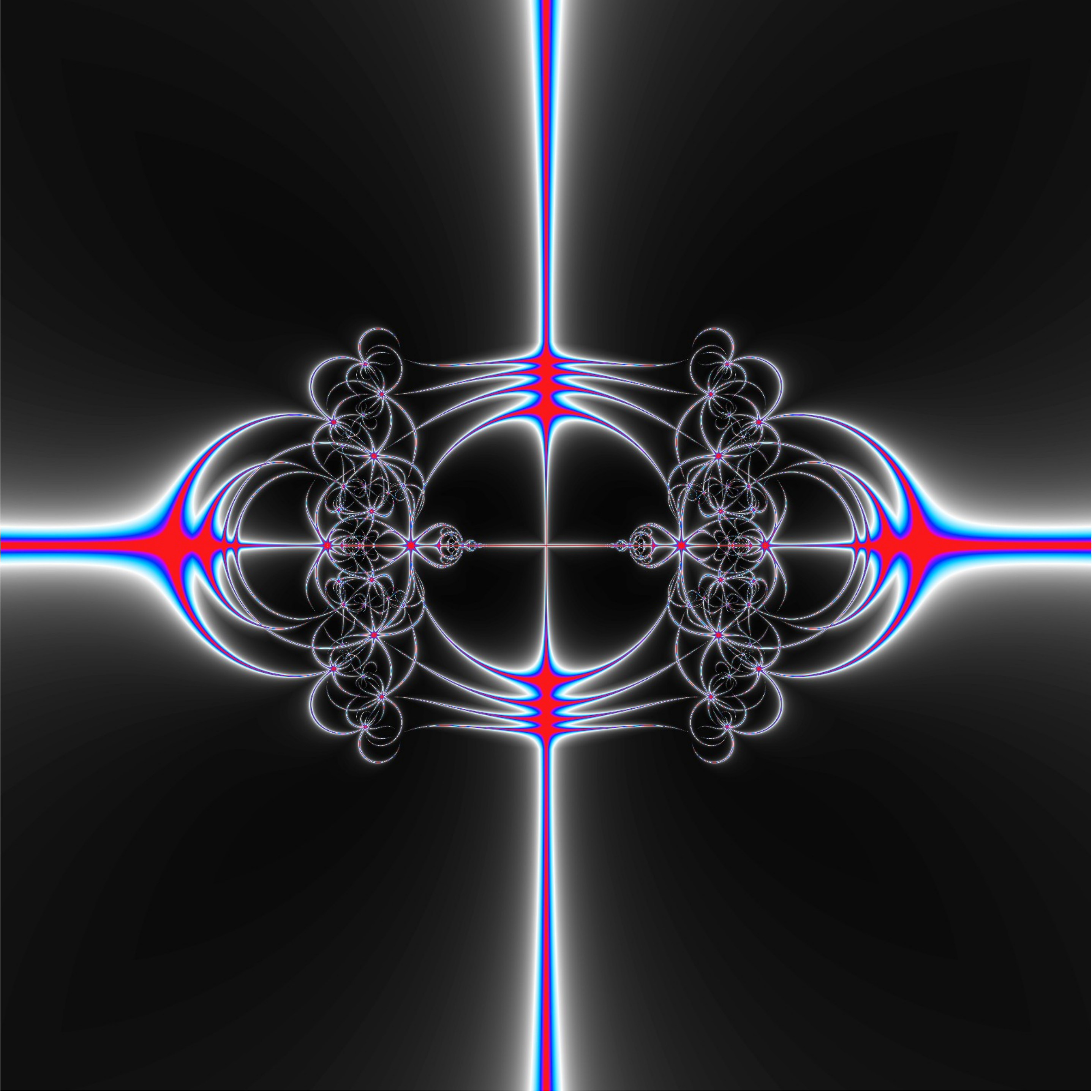
The function

$$z' = C\frac{1}{z^2 + 1}$$

generates the lovely epsilon spike image shown opposite.

The modulus argument produces the images below charting the distance from the origin after 8, 16 and 100 iterations respectively.



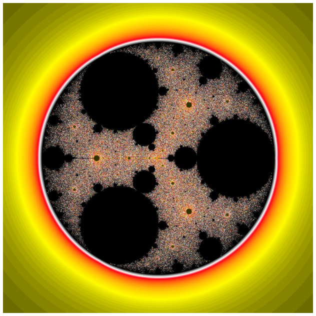


Final Fling

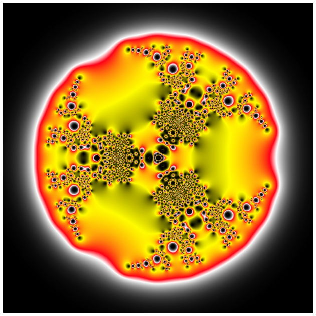
To finish this section on Mandelbrot fractals I present the Mandelbrot set for the function $\dfrac{z^3 + 1}{z^2} + C$.

Illustration A the standard Mandelbrot set. All points outside the unit circle escape to infinity. Inside the circle are three stable areas whose shape is that of the familiar Mandelbrot set for $z^2 + C$. But the rest of the disc is filled with a mass of chaotic points which seem to contain a number of isolated stable brots and circles. To what extent these features are real or artefacts of the method of computation I do not know.

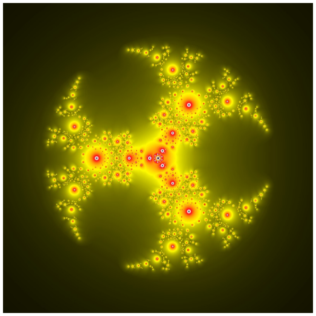
Illustration B shows the escape distance after 8 iterations. Not surprisingly it shows many similarities with illustration B on page 54. C is a plot of upsilon dots and opposite is the epsilon spike algorithm using C squared and inverted



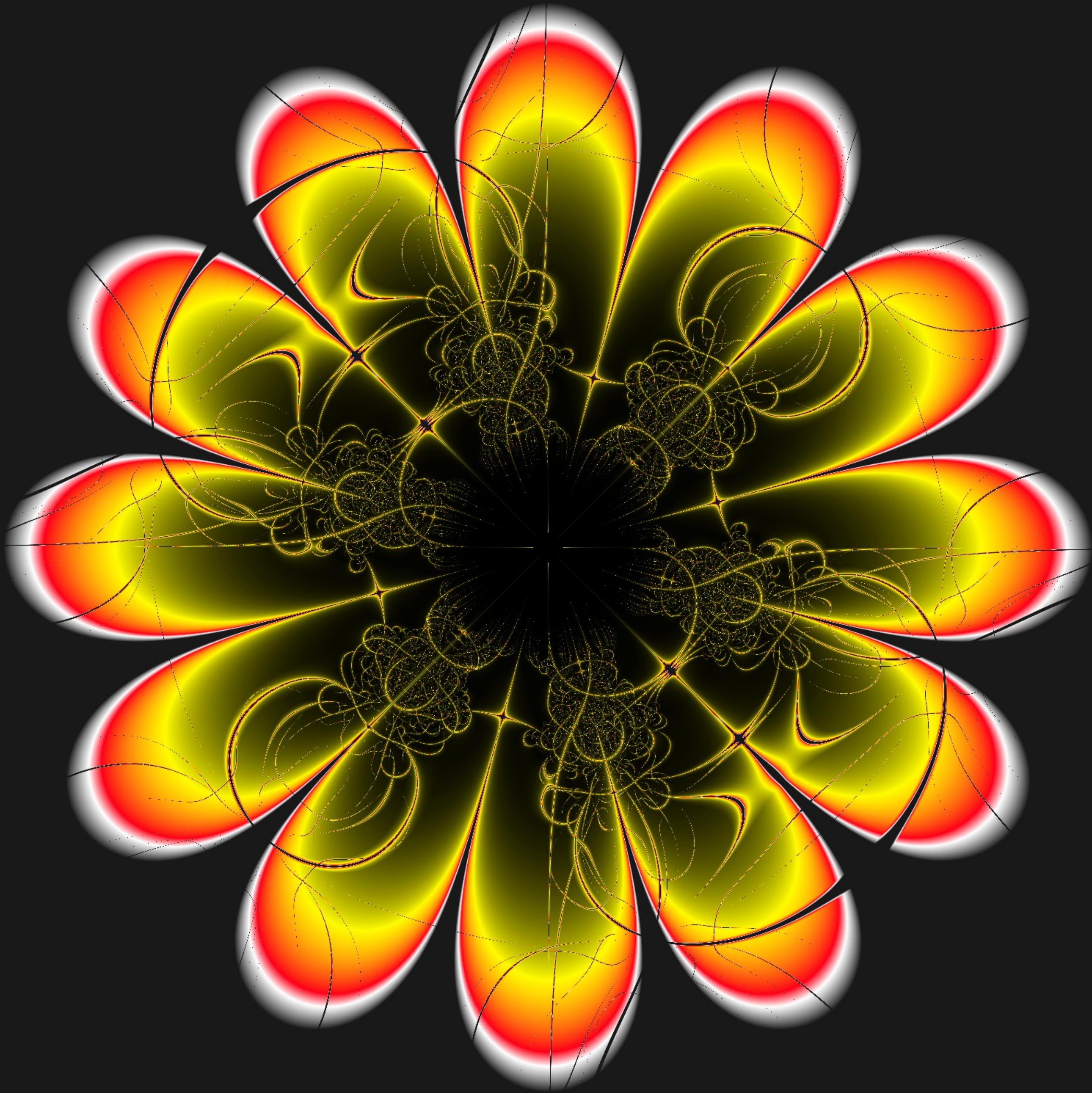
A



B



C



Classic Julia Sets

So far we have concentrated on the Mandelbrot set of a particular complex function. As I explained on page 2, this is generated by always starting at a fixed point (often (0, 0)) and exploring the effect of varying the constant *C*. Julia sets, however, are generated by fixing *C* and starting from any point in the complex plane. This means that, while there is only one Mandelbrot set for the function $z' = z^2 + C$, there are an infinite number of Julia sets[3].

The images opposite show the Julia sets of the above function for the following values of *C*: (0.27, 0), (0.41671875, 0.21015625), (-0.235, 0.654) and (-0.75, 0.077). The first is very close to the cusp of the cardioid.

The second is inside one of the small minibrots attached to lobe number 6 – hence the six armed stars. Since this value of *C* is actually inside a minibrot, the centre of the Julia set is stable and coloured with the lake colour (red). The third is close to one of the smaller lobes near lobe 3 while the last is in what is known as 'sea horse valley' – the crack between lobes 1 and 2.The variety of shapes is truly amazing.

All these Julia sets have rotational symmetry of order 2 because $(-z)^2 = z^2$. Interestingly, the function $z' = z^2 + az + C$ does not have rotational symmetry about the origin but it does have rotational symmetry about the point *a*/2 because the equation can be reduced to the form $z_1' = z_1^2 + C_1$ where $z_1 = z + a/2$ and $C_1 = C - a^2/4$.

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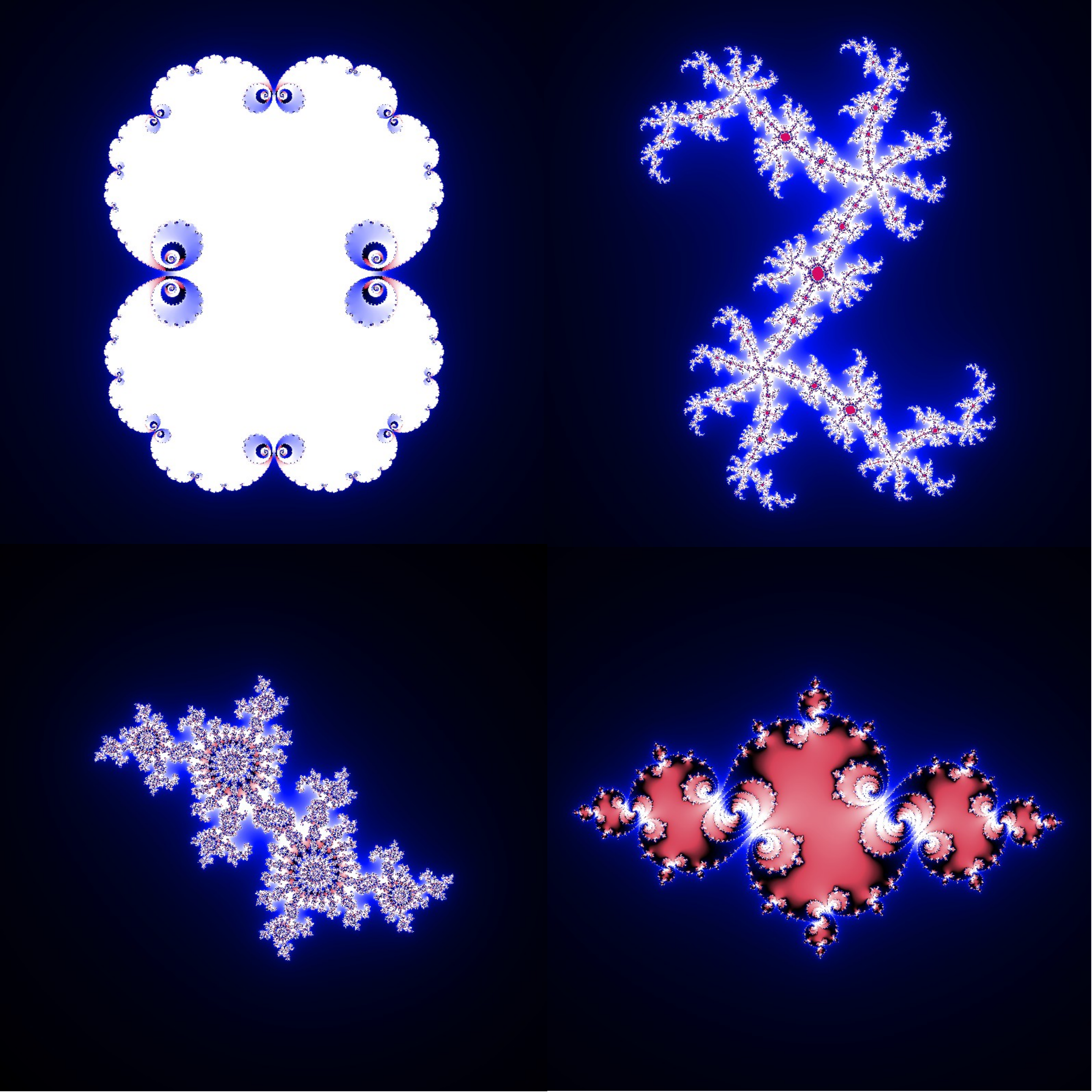
3 I am using the word 'set' here very loosely to mean the image generated by the stated algorithm rather than the formal mathematical definition of the word.

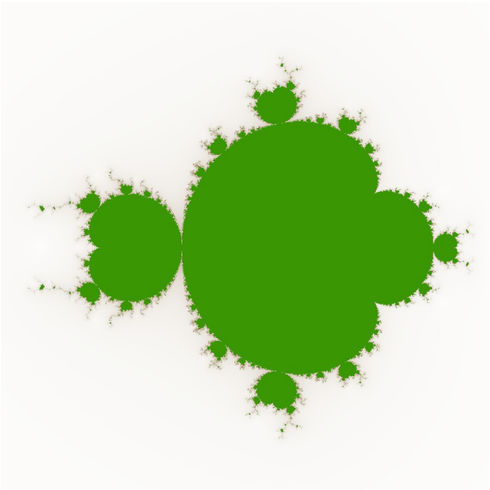


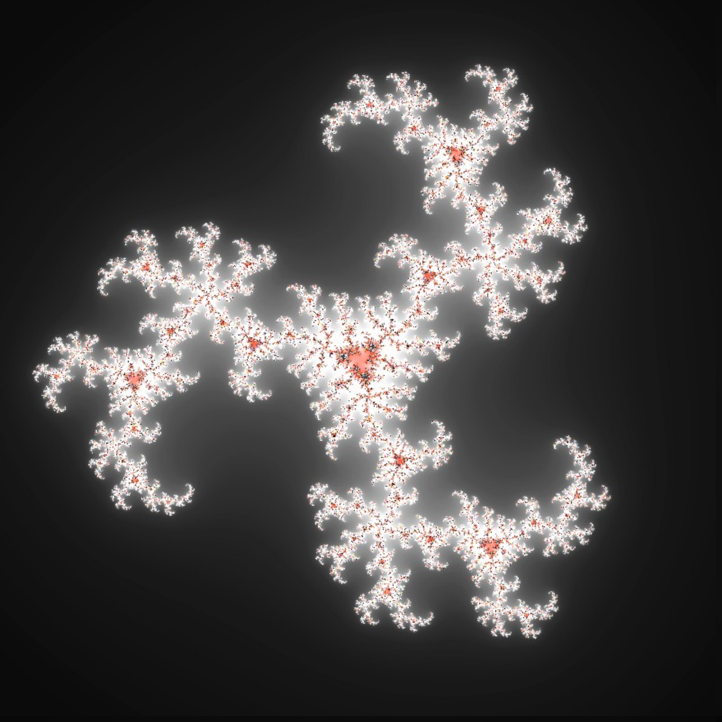
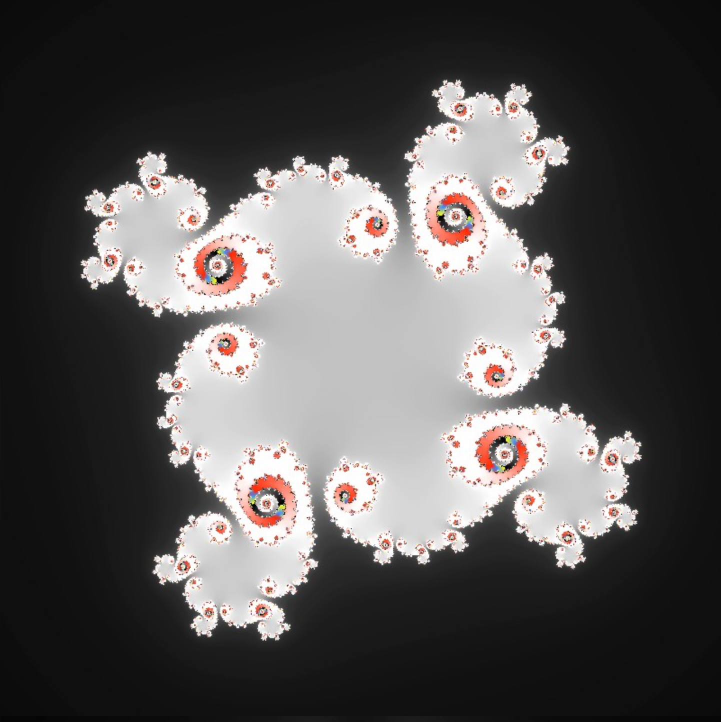
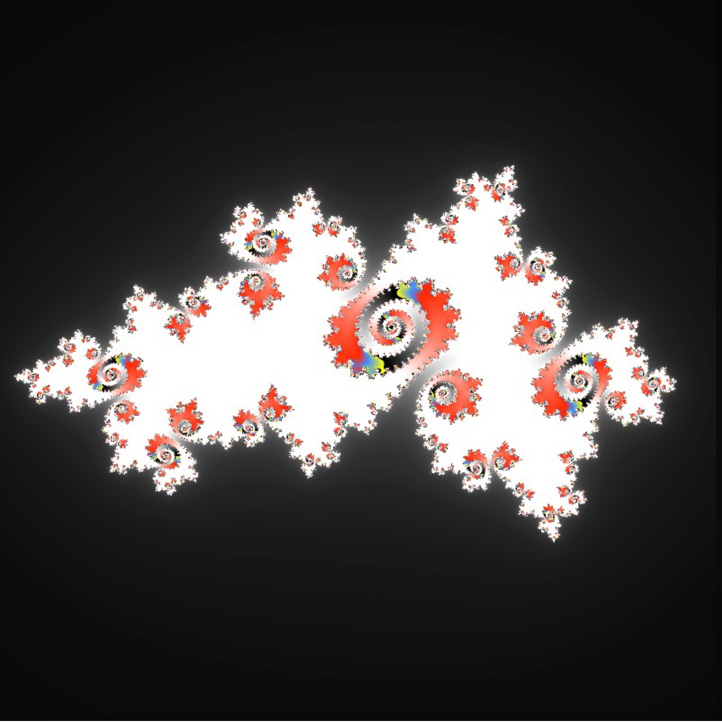
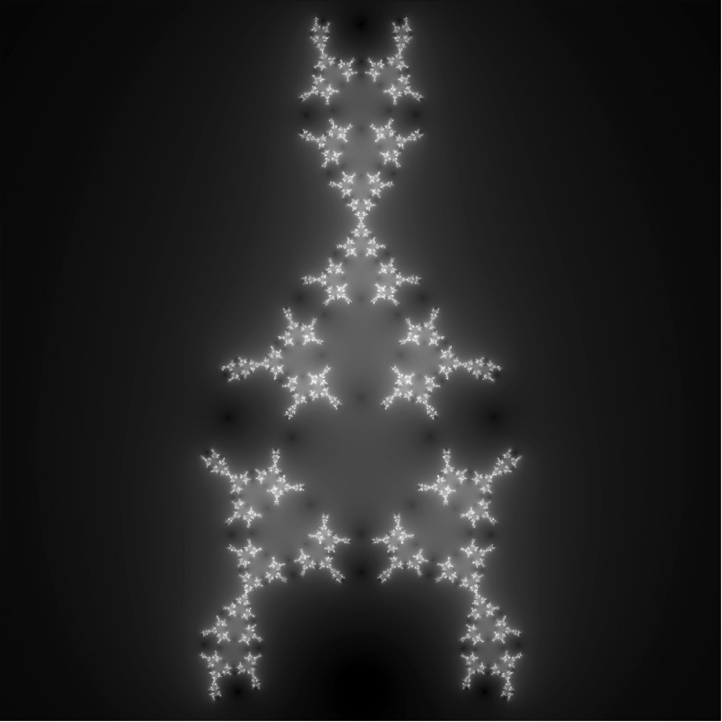
Polynomial Julia Sets

It will be no surprise to hear that while classical Julia sets have rotational symmetry of order 2, the Julia sets generated by cubic and quartic equations have symmetries of order 3 and 4 as shown in the upper two illustrations on the right.

The image at the bottom left, however, has no symmetry at all. It was generated by the formula $z' = z^3 - z + C$ with $C = (0.27, 0.28)$

The bizarre structure at the bottom right was generated by the formula $z' = \frac{z^3}{z - 1} + C$ with $C = (-1.85, 0)$ (rotated clockwise by 90°). The Mandelbrot set for this function is illustrated below and the value of C chosen lies on the X axis between the 'claws' on the left.



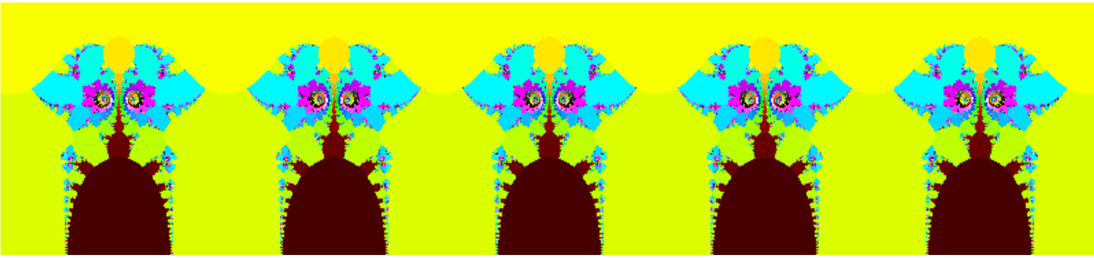
Exponential Julia Sets

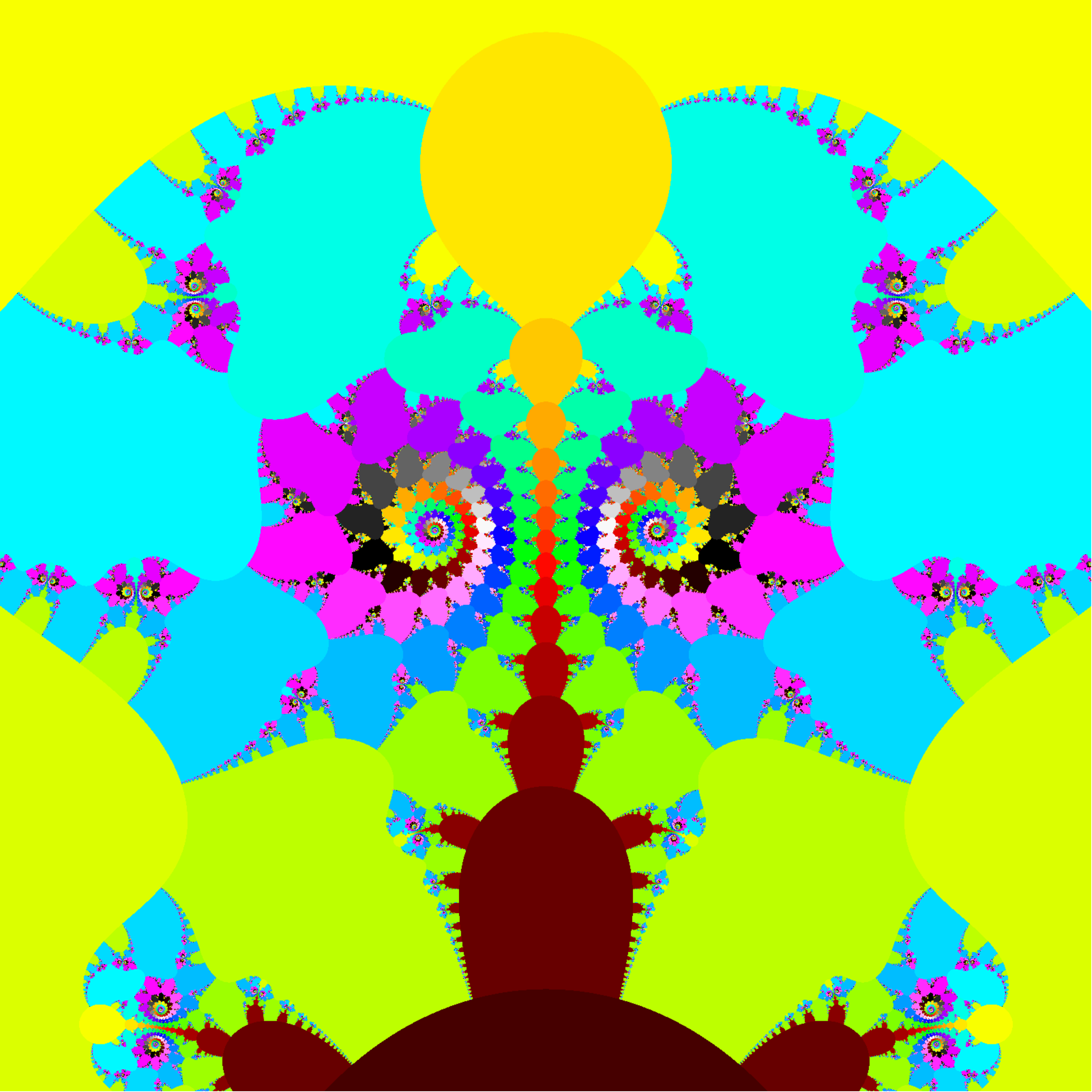
The function $z' = C\, e^{z}$ does not have a meaningful Mandelbrot set because for all values of C other than zero, all starting points diverge to infinity.

On the other hand, its Julia sets are very interesting because for any given value of C, different starting points will diverge at different rates. As an example, consider the case when $C = 0.4$. It is not difficult to show that all starting values of z diverge to infinity, even negative ones. But some starting values of z cause the point to go round and round in circles, gradually spiralling out until it reaches the limit fixed by the program.

The image below has been rotated clockwise so the imaginary axis is horizontal and negative real numbers are at the top. The centres of the spirals repeat along the imaginary axis with a period of 2π.

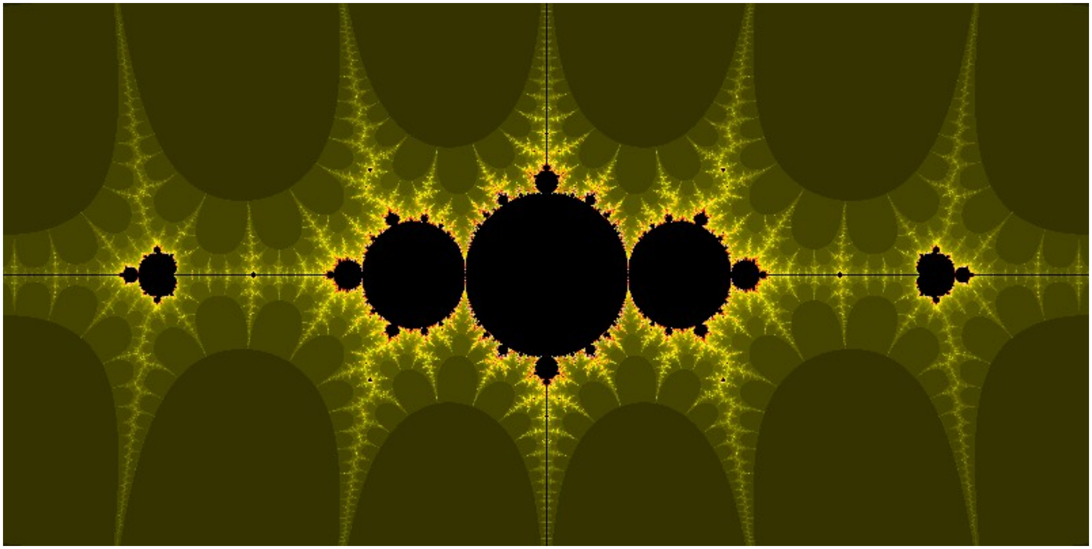
The rather 'blocky' nature of the image opposite is due to the fact that if the point just manages a complete circuit before exceeding the fixed limit, it has to go round a complete revolution before escaping next time round.





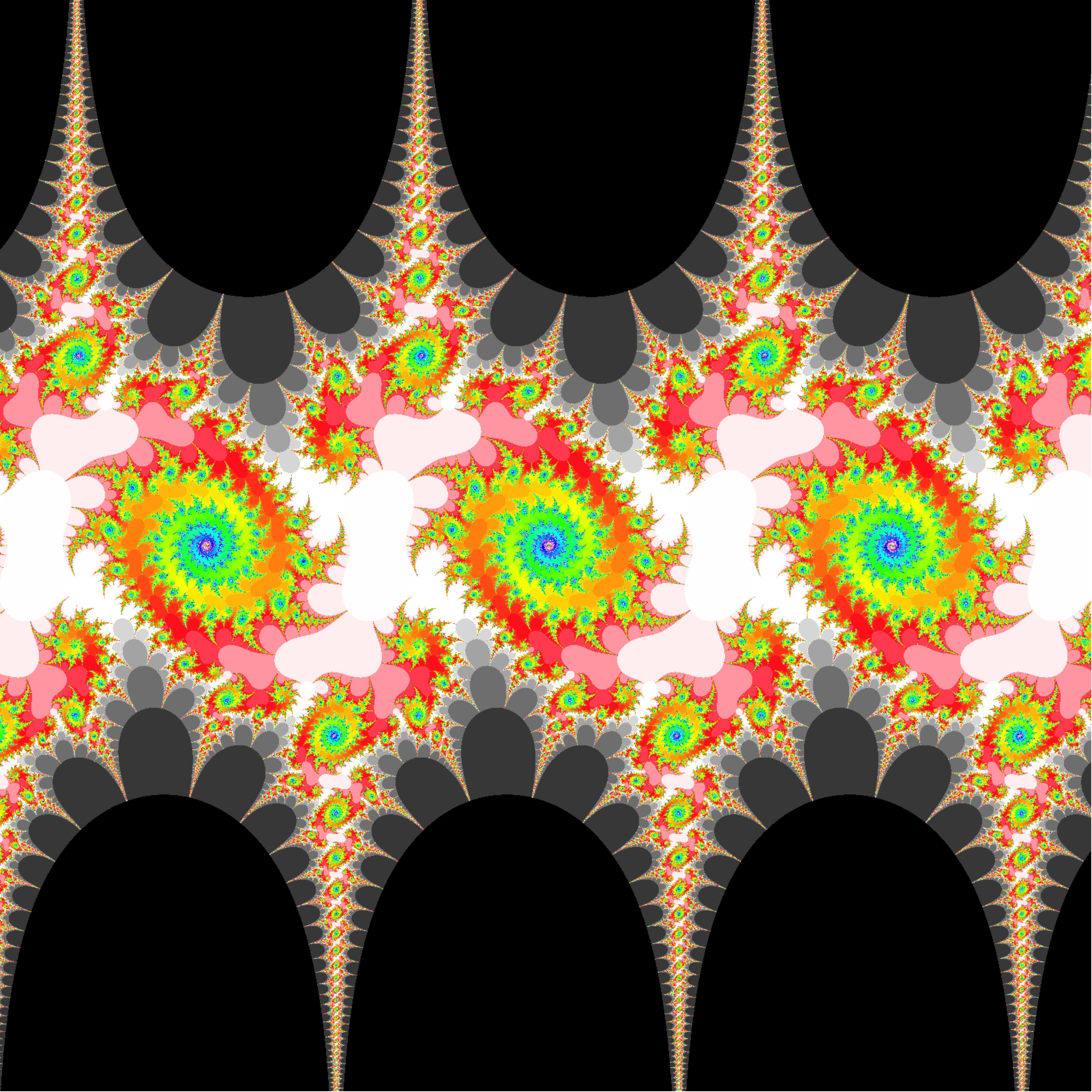
Transcendental Julia Sets

The function $z' = C \sin(z)$ has a perfectly respectable – if rather surprising – Mandelbrot set shown below.



The central circle has a radius of 1 and the smaller circles on either side are centred on the point (π/2, 0). On each side there are a series of minibrots the largest of which are centred on ($n \pm \pi/2$, 0). As usual, interesting Julia sets are to be found when C is near the edges of the stable regions.

The illustration opposite has $C = 1 + 0.4i$

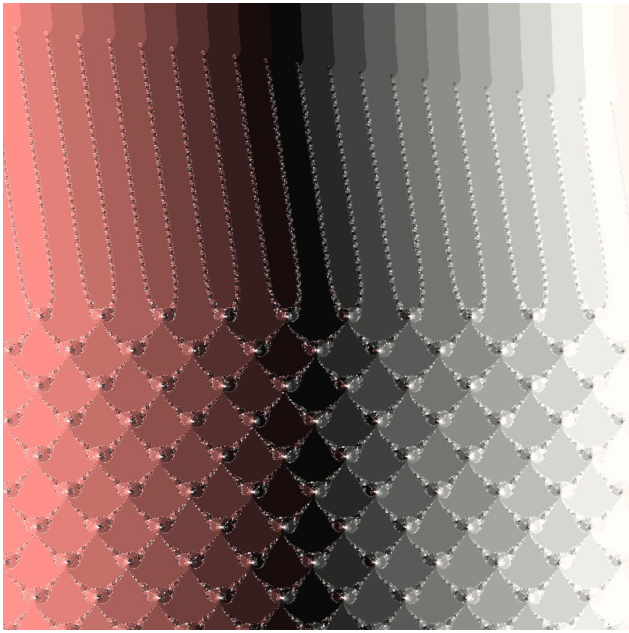


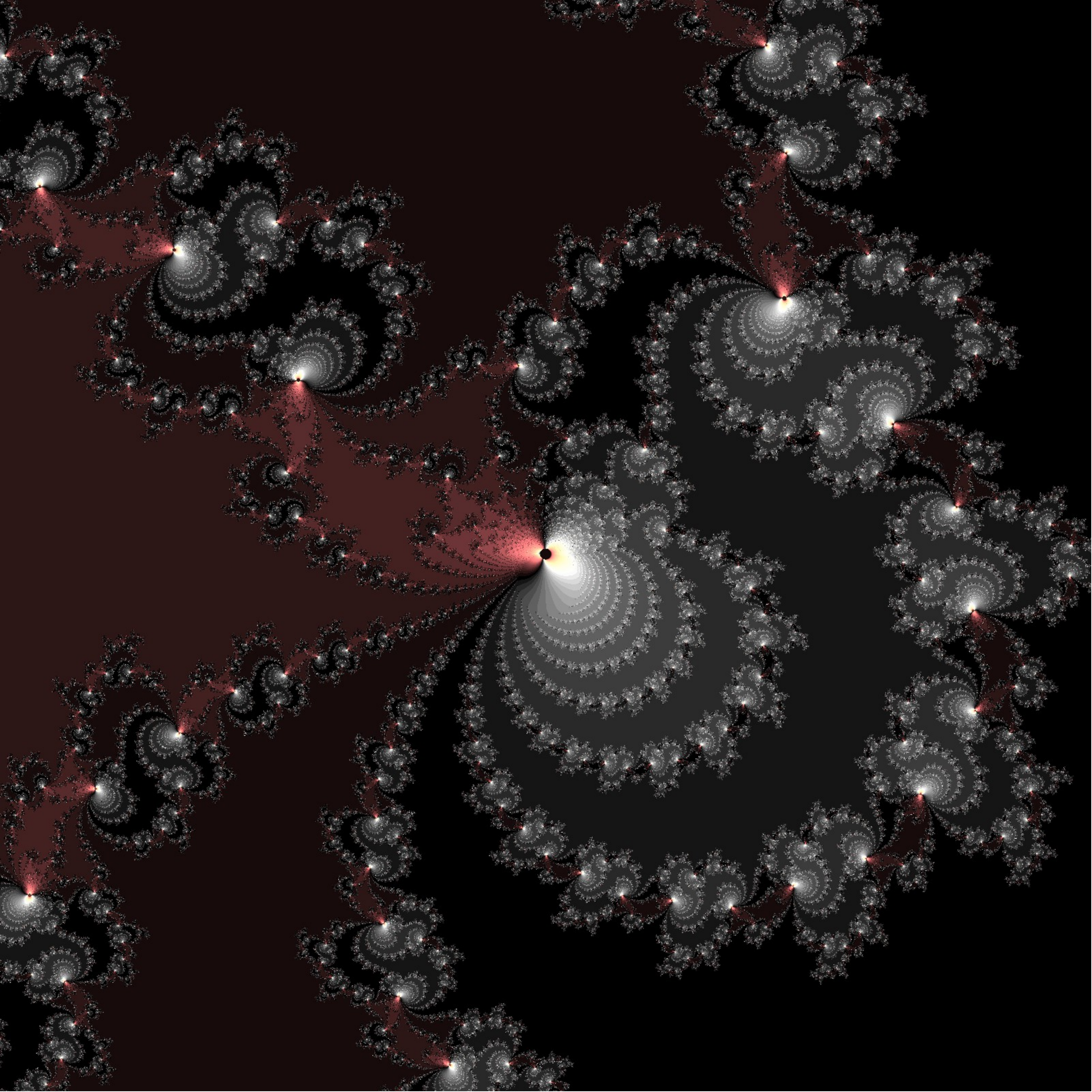
A Tangent Julia Set

Illustrated opposite page 22 is the Mandelbrot set of the function

$$z' = z - \tan(z) + C$$

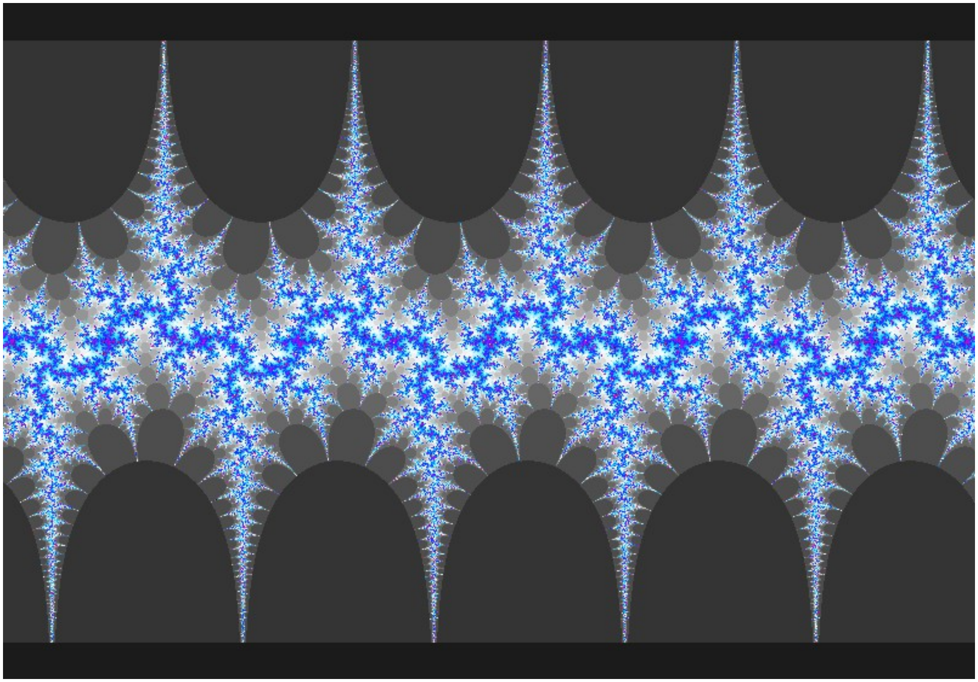
in which I pointed out certain areas of chaos. Inside these areas there are islands of partial stability and values of C within these areas generate remarkable Julia sets. The illustration below shows part of the Julia set for the point (1.6, 0.8) and opposite is a detail of one of the 'bosses' in the negative half of the plane.

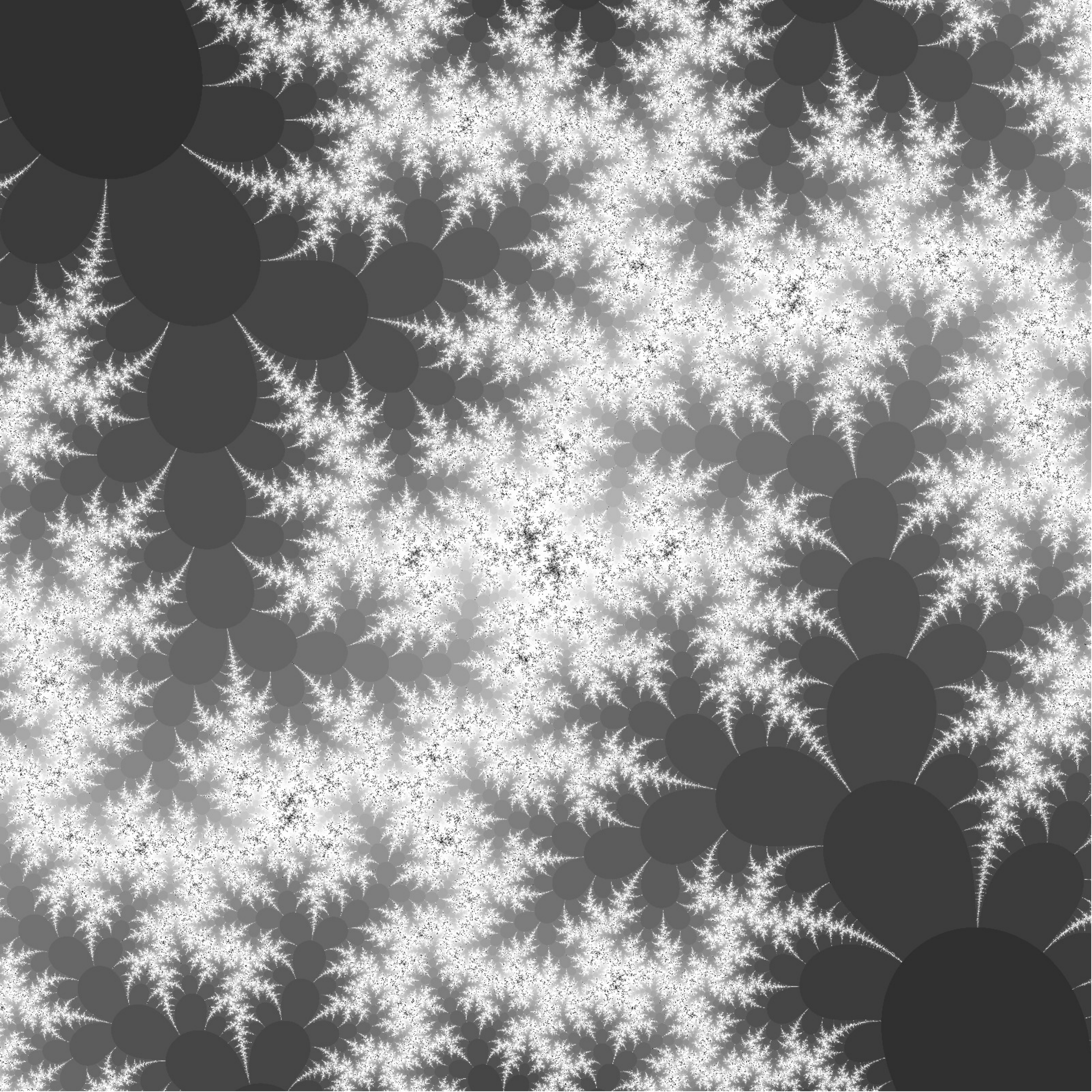




A Hyperbolic Julia Set

A typical Julia set generated by the function $z' = C \cosh(z)$ is not dissimilar to that generated by the sin function (except that it repeats along the imaginary axis, not the real one). In detail, however it reveals a wonderful array of intricate patterns which resemble the growth of frost on a window pane.





Polynomial Fractals

All the algorithms we have examined so far involve a single expression in a complex variable. It is this fact which gives these fractals a degree of coherence which makes them attractive. However, it is not necessary for there to exist a simple complex equation describing how a point (x, y) is transformed into a new point (x', y') – a pair of equations will suffice such that

$$x' = f(x, y) \quad \text{and} \quad y' = g(x, y)$$

If f and g are polynomials, then the resulting fractal (if it exists) may be called a polynomial fractal.

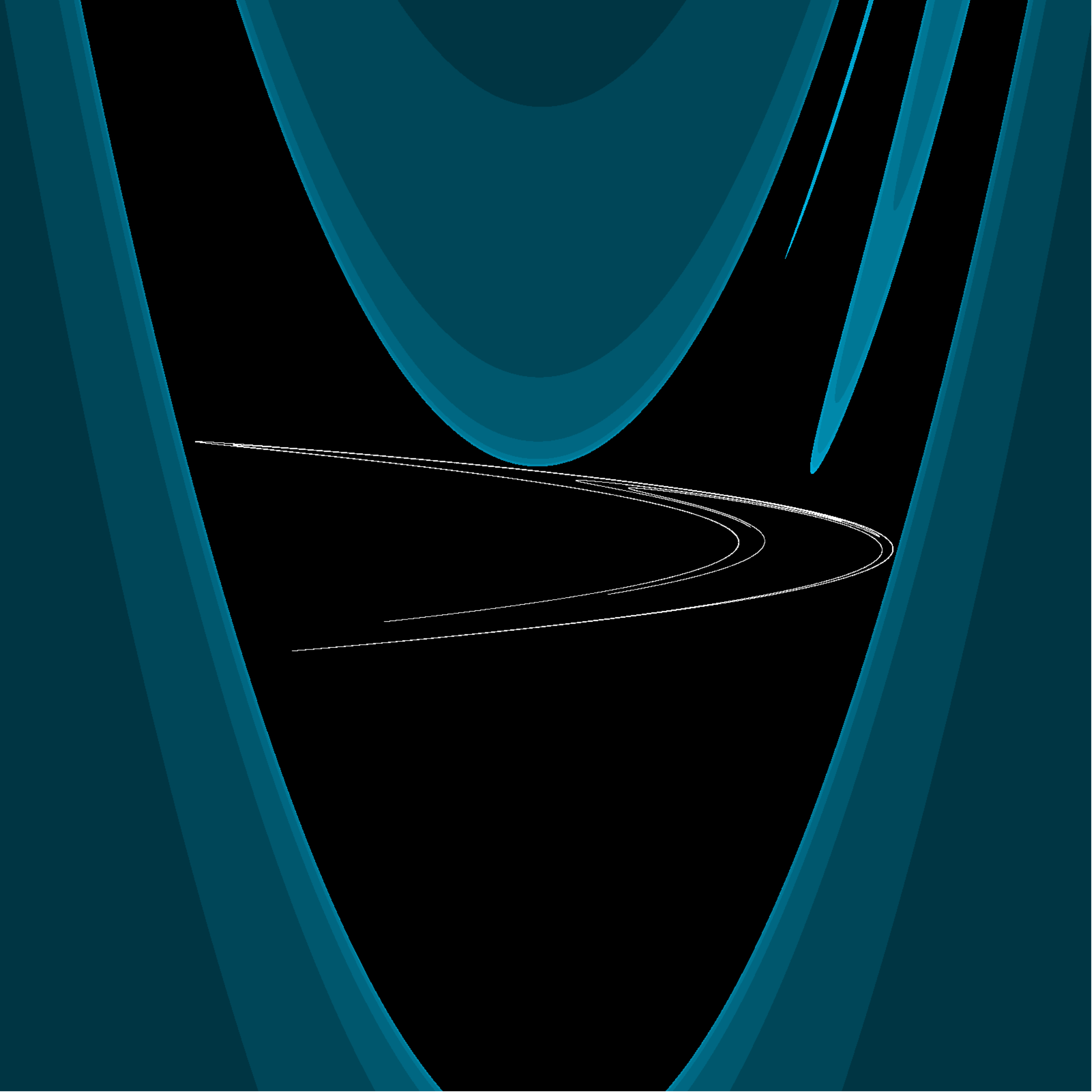
In general, any initial point (x_0, y_0) can end up doing one of three things:

1. it may escape to infinity
2. it may home in on a single point or a finite periodic cycle of points
3. it may home in on what is called a 'strange attractor' – this is an infinite set of points which are strictly confined to a small region of the plane, often displaying a fractal structure.

One of the first strange attractors to be discovered was the Hénon attractor whose algorithm is

$$x' = 1 - 1.4x^2 + y$$
$$y' = 0.3x$$

In the image opposite the basin of attraction is shown in black with colours indicating the rate of escape as in a standard Julia set image.



More Strange Attractors

Shown opposite are four of my favourite strange attractors together with their basins of attraction.

First (top left) is the Tinkerbell attractor whose equations are

$$x' = x^2 - y^2 - 0.6\mathrm{y}$$
$$y' = 2\mathrm{x}(y + 1) + 0.5\mathrm{y}$$

Top right is what I call the tulip attractor with the following particularly simple equations:

$$x' = xy$$
$$y' = x^2 - 1$$

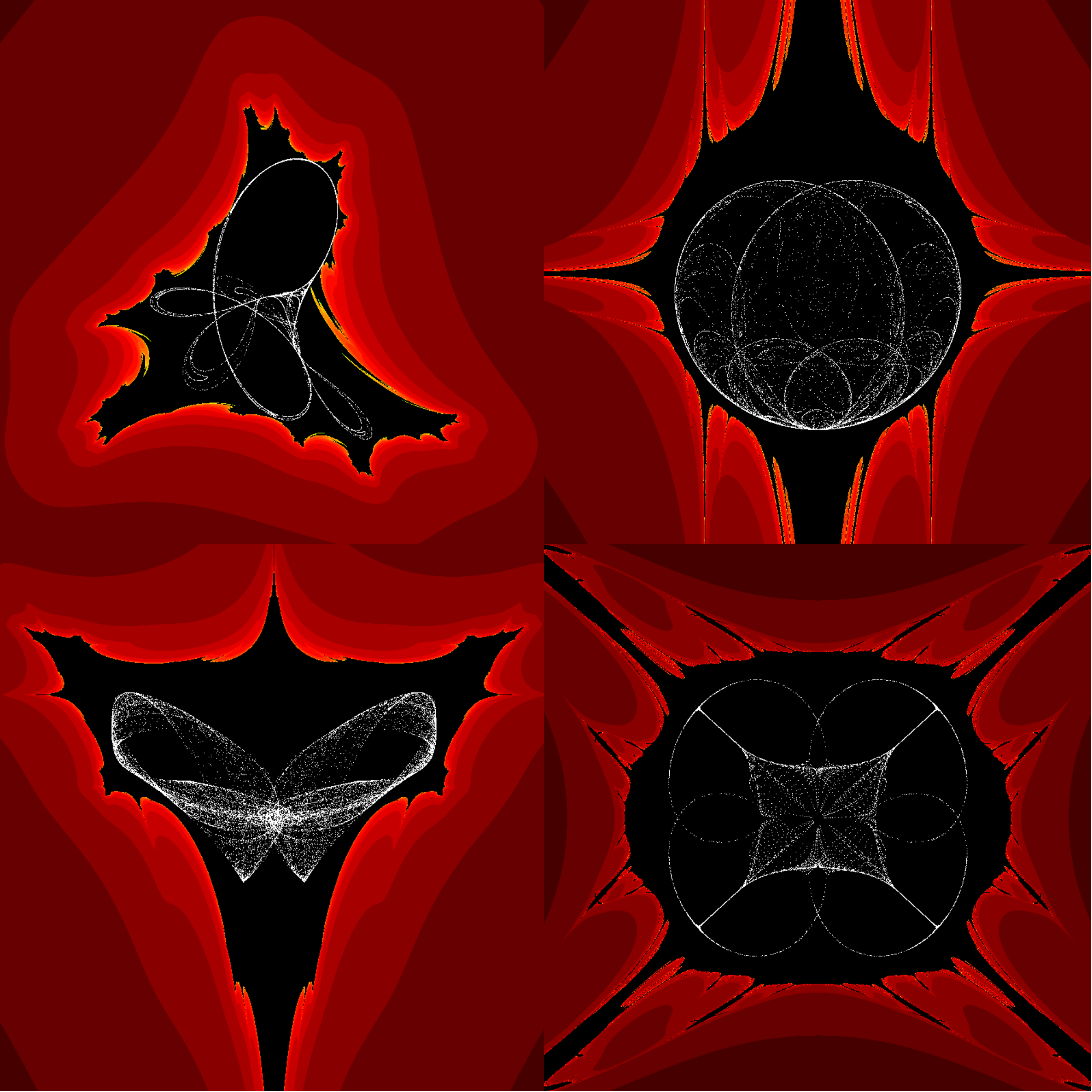
Bottom left is the bow tie attractor:

$$x' = 1.8\mathrm{x} - 2\mathrm{xy}$$
$$y' = x^2 - 0.5\mathrm{y}^2 - 0.7\mathrm{y}$$

and finally, bottom right is the spider's web attractor:

$$x' = x(1.2 - 1.1\mathrm{x}^2 + y^2)$$
$$y' = y(1.2 + x^2 - 1.5\mathrm{y}^2)$$

(The last three were discovered by the author using a search program after an idea by J.C.Sprott)



Barry Martin Structures

These amazing structures (they are not fractals in the true sense of the word) were discovered by Barry Martin in 1986 and even now, nobody appears to know exactly how they are produced.

The starting point is a simple hopalong formula which, in general, produces a simple ellipse from any given starting point:

$$x' = y + bx$$
$$y' = -x$$

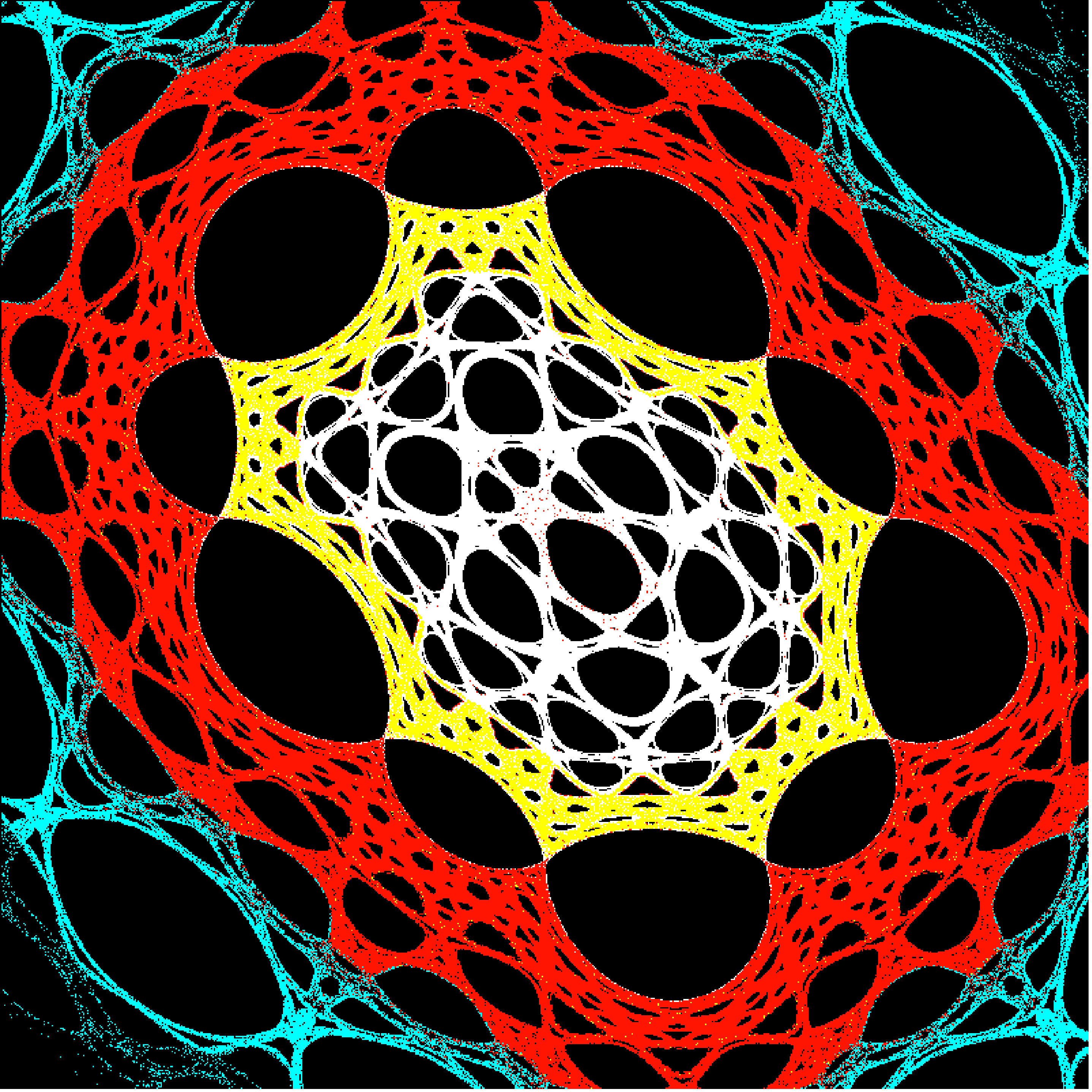
If $b = 0$ then any starting point (p, q) simply moves through a sequence of four points: $(p, q) \rightarrow (q, -p) \rightarrow (-p, -q) \rightarrow (-q, p) \rightarrow (p, q)$. But if b is a small number, the sequence does not quite end exactly where it started and the result is an apparently continuous ring of points forming an ellipse. (I say apparently because, for all I know, there may be values of b which result in a periodic cycle but I haven't been able to find any.)

What Barry Martin did was to replace the function bx with a variety of different functions of x. The classic Barry Martin fractal shown on the page opposite is generated using the formula

$$x' = y + \operatorname{sgn}(x)\sqrt{|ax - b|}$$
$$y' = 1 - x$$

where $a = 4$ and $b = 0.5$. (The curious function in the equation for x' is a sort of signed square root. The values of a and b are not critical.)

The extraordinary thing about this fractal is the way it goes on growing, seemingly without limit.



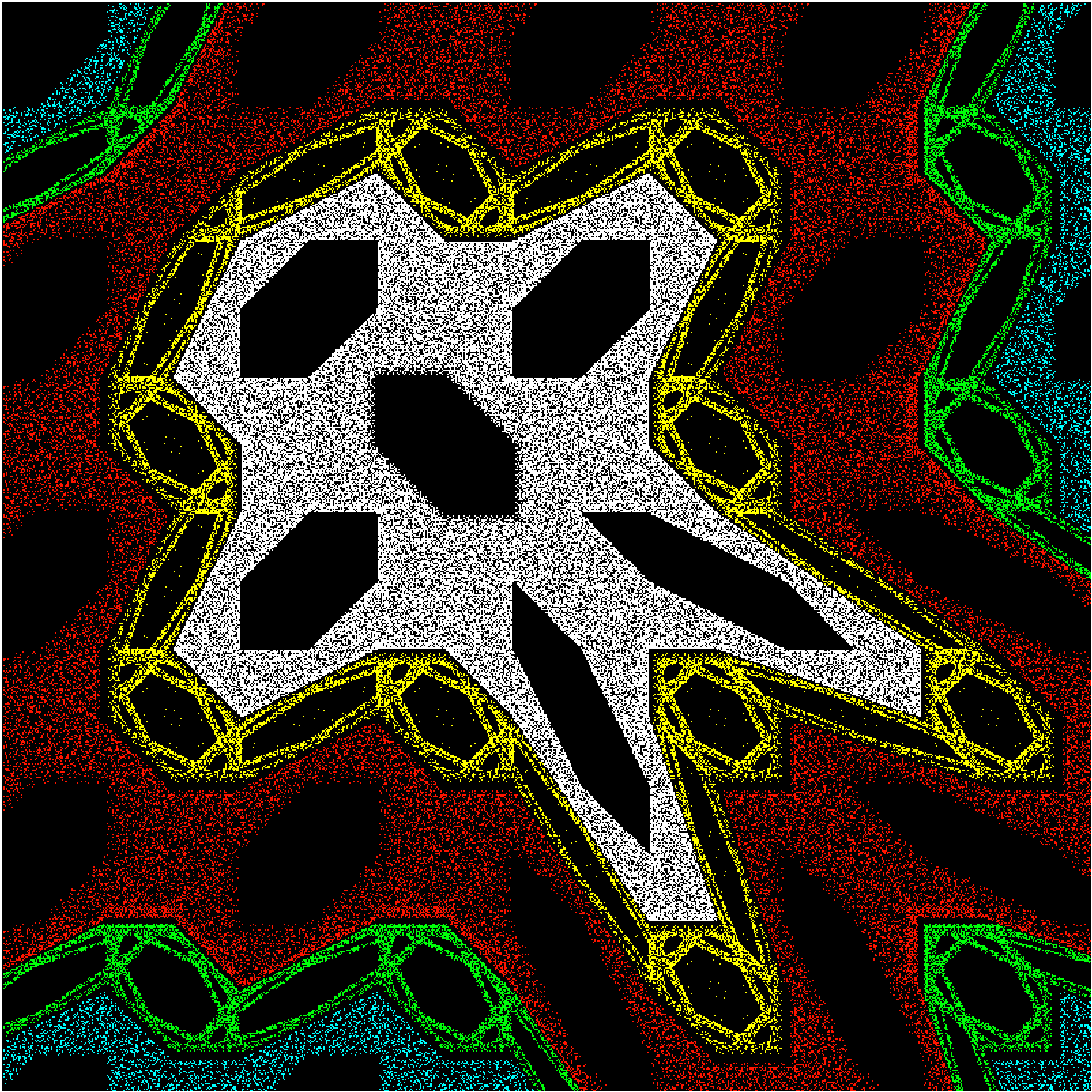
The Gingerbread Man

This extraordinary structure is generated from a particularly simple formula:

$$x' = y + |x|$$
$$y' = 1 - x$$

In the centre of the gingerbread man there is a black lozenge which stretches from (0, 1) at the top left to (2, -1) at the bottom right. (The centre of this lozenge is not the origin; it is at (1, 0).) It is not difficult to show that *any* starting point in this region cycles through just 6 points. For example, starting at (1, 0.5) we cycle through (1.5, 0), (1.5, -0.5), (1, -0.5), (0.5, 0), (0.5, 0.5) and back to (1, 0.5). Similarly, any starting point in the man's head (e.g. (-1, 2) cycles through his arms and legs with a period of 30. In fact all points within the black lozenges are stable with different periodicities which are multiples of 30. It is this (possibly unique) property of this function that is responsible for this extraordinary structure.

Other starting points round the central figure generate bands pierced by black holes which contain further quasi periodic structures. These bands are of two types; some (e.g. those in red and blue opposite) appear quite solid – like the gingerbread man himself; others (in yellow and green) are doubly pierced with black holes.



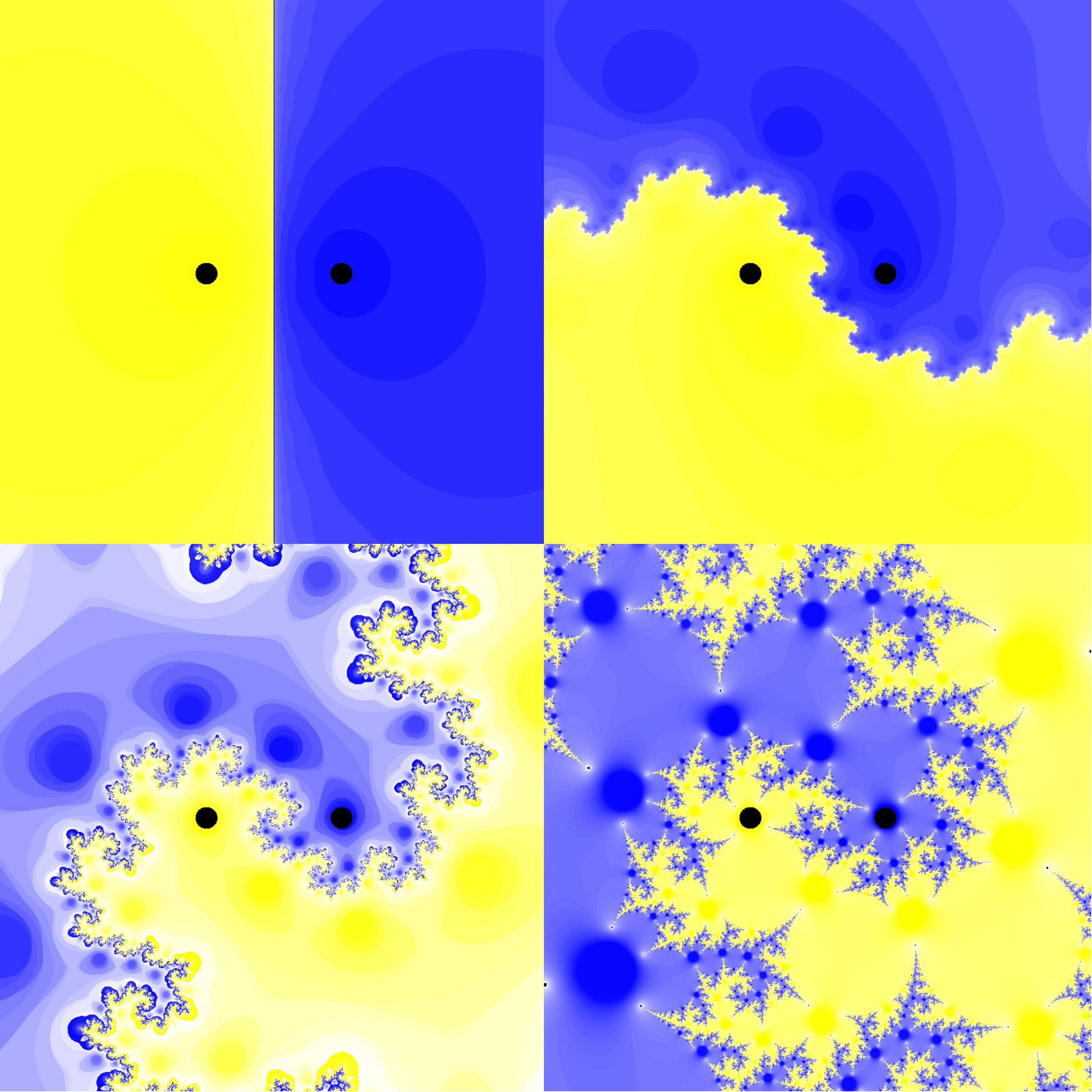
Newton-Raphson Fractals

The newton-Raphson method is a method for finding the roots of an equation – that is to say, a way of finding values of *z* which make the function zero. I do not need to describe the method here in detail; suffice it to say that you start with a point, hopefully somewhere near to one of the roots, and use a derivative of the equation to calculate a 'correction' which, when subtracted from your original guess, moves you closer towards the desired solution.

For example, if you want to find the roots of the equation $z^2 = 1$ the correction turns out to be $\frac{z^2 - 1}{2z}$. What we do, therefore is to iterate the equation $z' = z - \frac{z^2 - 1}{2z}$ until the correction reduces to near zero.

Now the roots of the equation $z^2 = 1$ are, of course, 1 and –1 so it will not surprise you to learn that any starting point whose real part is greater than 0 homes in on the positive root and any point whose real part is negative homes in on the negative root. In other words the boundary between the two zones of attraction is simply the imaginary axis.

In order to generate an interesting fractal, we make life a little more difficult for the algorithm by first multiplying the correction by a complex number R. When R = 1 the boundary is a straight line as shown in the first illustration opposite. The other illustrations are for R = (1 + 0.5i), (1 + 0.9i) and (1 + 0.995i). (The two black dots are, of course, the two roots.)



The Cube Roots of 1

The equation $z^3 = 1$ has three complex roots, 1, –½ + √3i and –½ – √3i.

It turns out that the correction term we need is $\frac{z^3 - 1}{3z^2}$ and the function we therefore need to iterate is

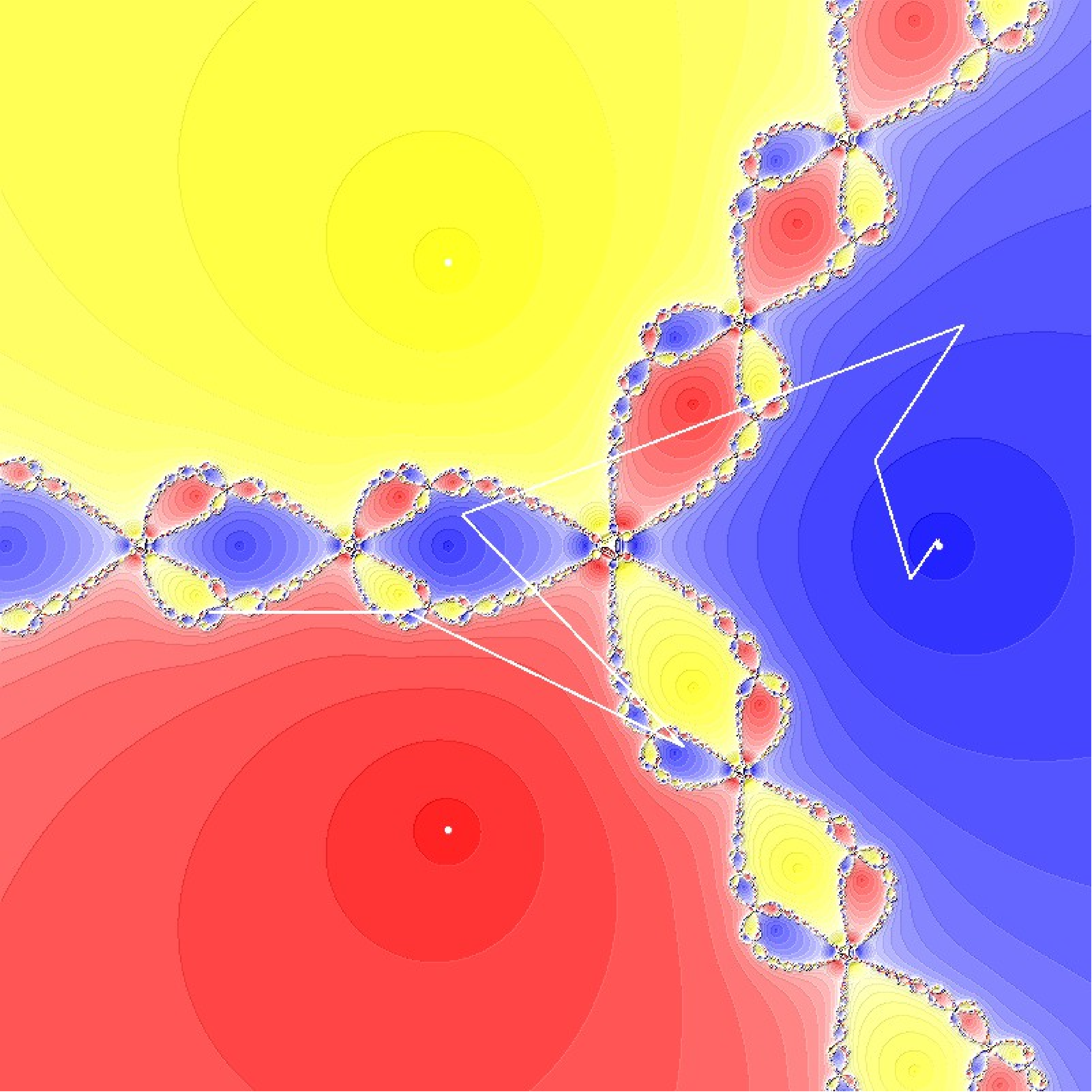
$$z' = z - \frac{z^3 - 1}{3z^2}$$

Now you might expect that starting points would simply migrate towards the nearest root but this turns out not to be the case. Instead the boundary between the basins of attraction turns out to be a fractal best illustrated by colouring the three basins in shades of contrasting colours.

Obviously, if you start close to one of the roots, that is where you will end up. But if you start somewhere near the boundary between the large coloured regions, you might end up anywhere!

For example,the white line shows the path of *z* which starts closest to the red root – but ends up at the blue one!

The boundary between the red, yellow and blue zones has the fascinating property that every point on the boundary is a place where *all three* zones meet!



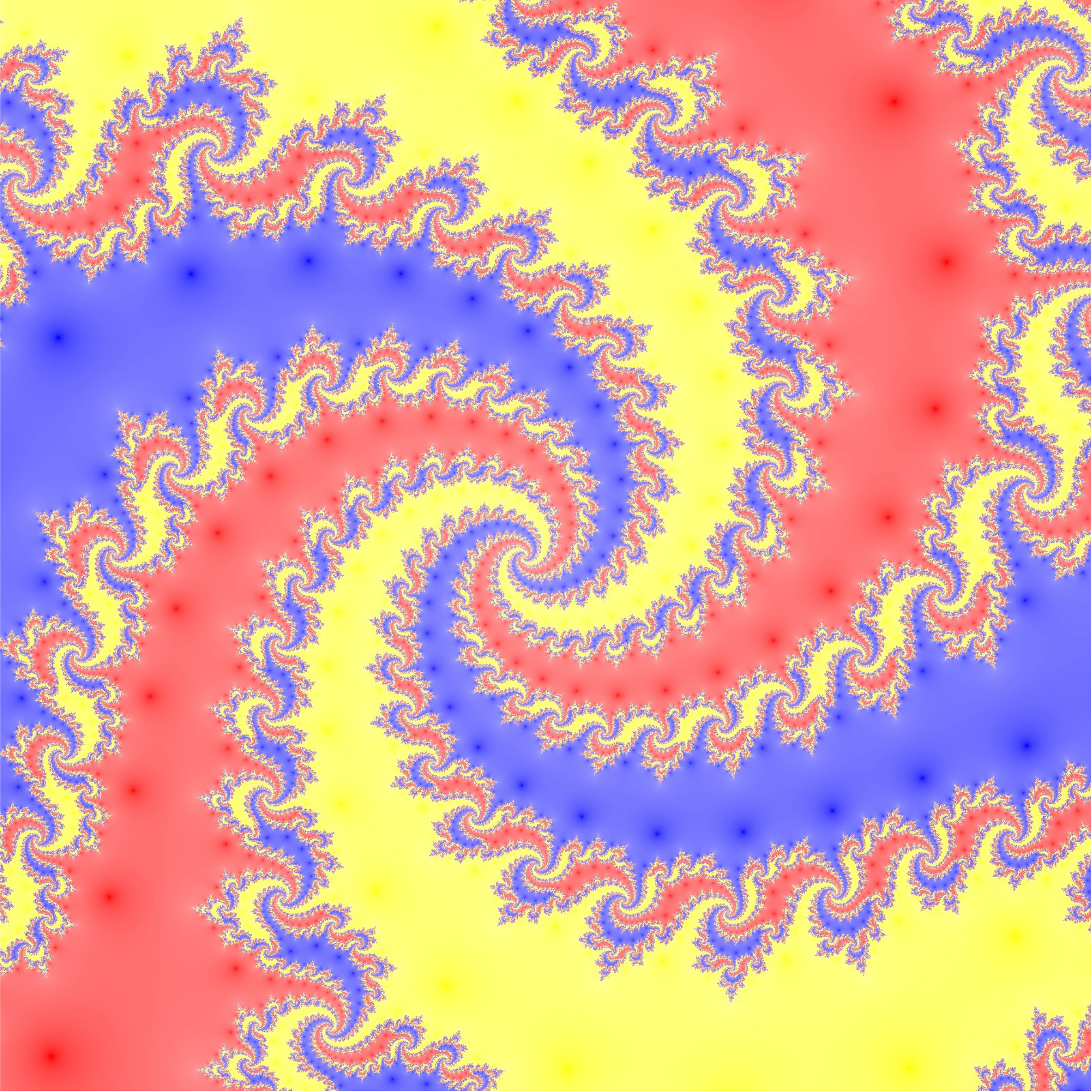
Making Life Difficult

Just as we did with the Newton-Raphson algorithm for the solution to $z^2 = 1$, if we multiply the correction factor by a complex number R – i.e. if we use the equation

$$z' = z - R\frac{z^3 - 1}{3z^2}$$

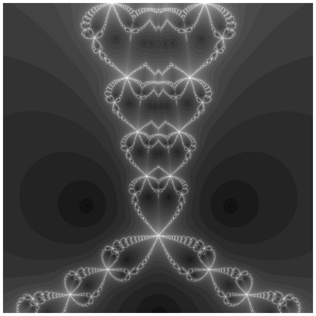
the triple chain illustrated on the previous page becomes more and more twisted.

For example, the high resolution image opposite shows the case where R = 1 + 0.95i.

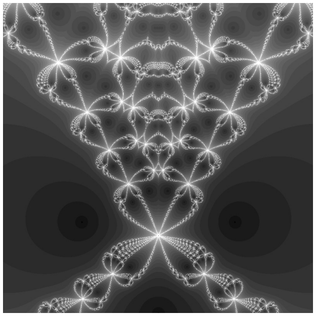


Christmas Lights

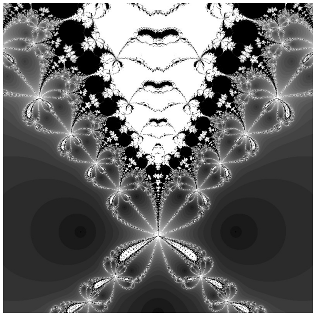
The images below (which have been rotated clockwise by 90°) were generated by using a variation[4] of the Newton-Raphson method to 'solve' the equation $z^n = 1$ with $n = 3.2$, 3.6 and 3.88. Now obviously an equation cannot have 3.6 roots any more than a mother cannot have 3.6 children. What happens is that the boundary which lies on the real axis (now vertical) expands and tears itself apart producing a wonderful array of swags and strings which are shown in detail in the illustration opposite. When $n = 3.88$ then the region in between the arms becomes more and more complex and some of the points within it take a very long time to decide which root to go to!



$n = 3.2$



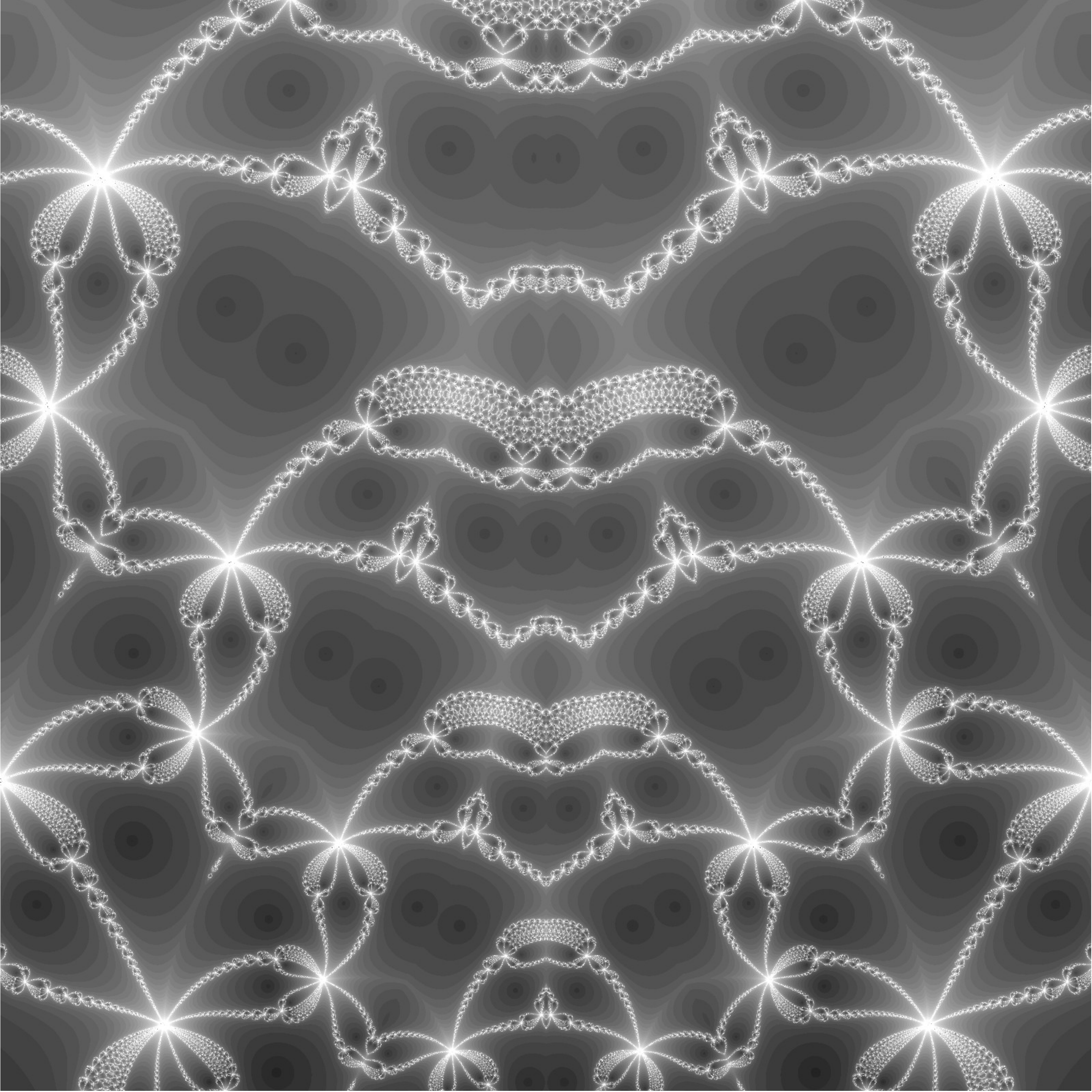
$n = 3.6$



$n = 3.88$

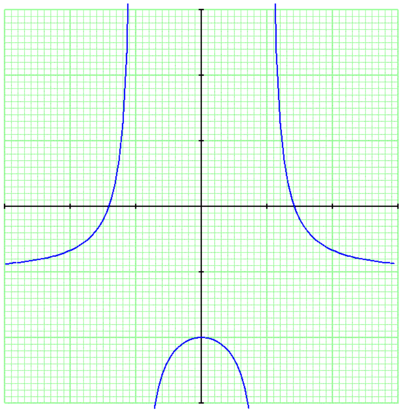
The illustration opposite shows a detail of the swags when $n = 3.7$

4 When n is fractional we have a slight problem. During the calculation it is necessary to calculate the argument (i.e. the angle) of a complex number. This is a many-valued function. If we choose the smallest positive value, the N-R routine homes in on the correct roots but the resulting fractals are not very pleasing. To generate the images on this page I have used values in the range $-\pi$ to $+\pi$.



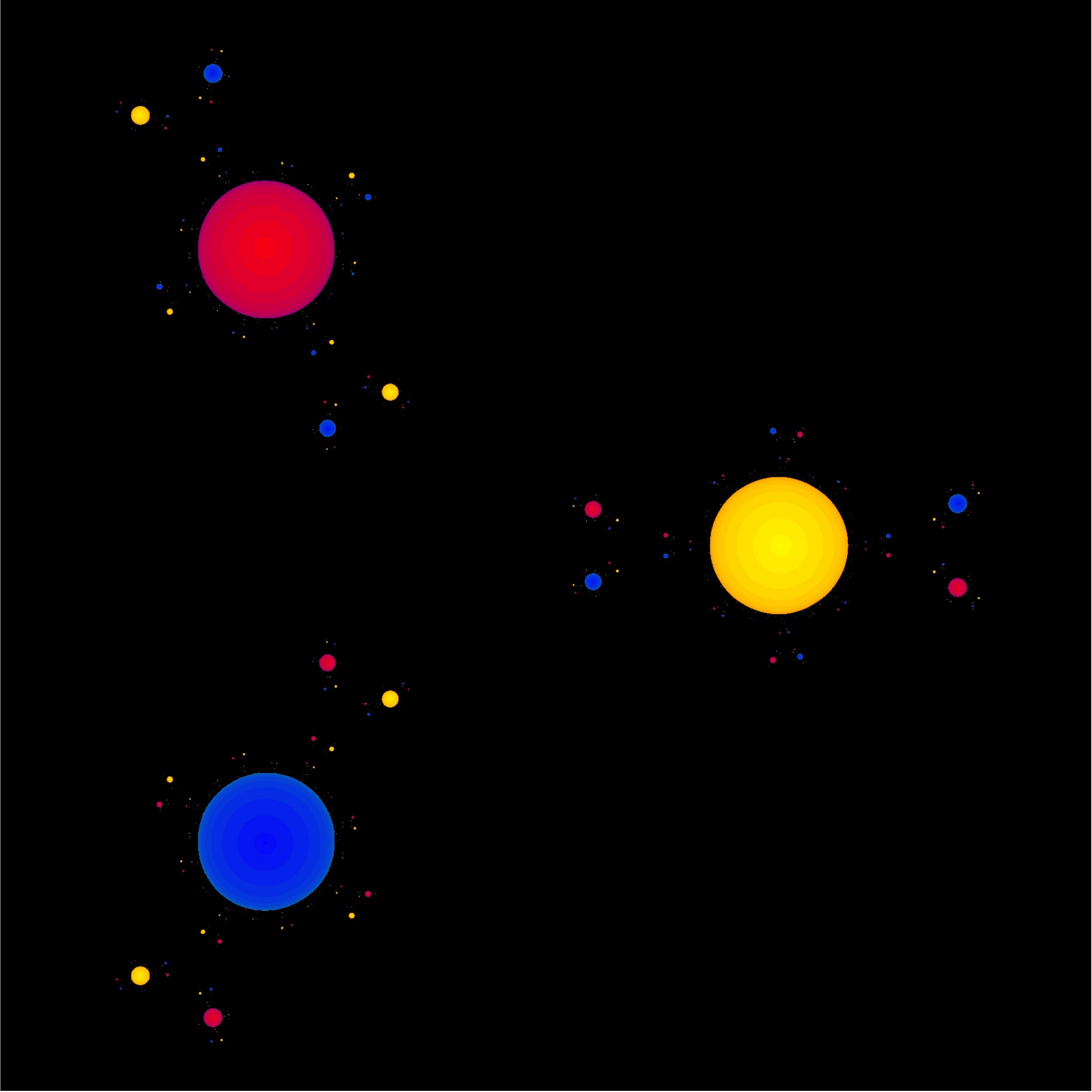
Problem Functions

Consider the equation $\frac{1}{x^2 - 1} = 1$. This can be simplified to $x^2 = 2$ whose roots are obviously $\pm\sqrt{2}$. But if we try to solve the equation using the Newton-Raphson method, we run into a problem because the function $f(x) = \frac{1}{x^2 - 1} - 1$ goes off to infinity at ± 1 as shown below.



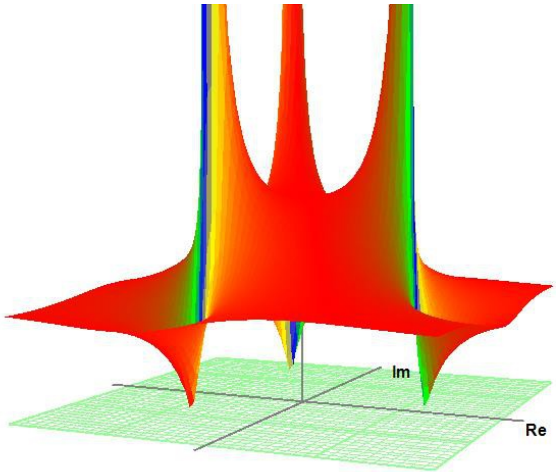
Even when using complex numbers, as long as we start quite close to one of these roots, all is well – but in general the majority of starting points diverge off to infinity.

In the illustration opposite the function used is the cubic function $f(z) = \frac{1}{z^3 - 1} - 1$ which has the familiar three roots and starting points which home on these roots are shown in colour. The black areas diverge.

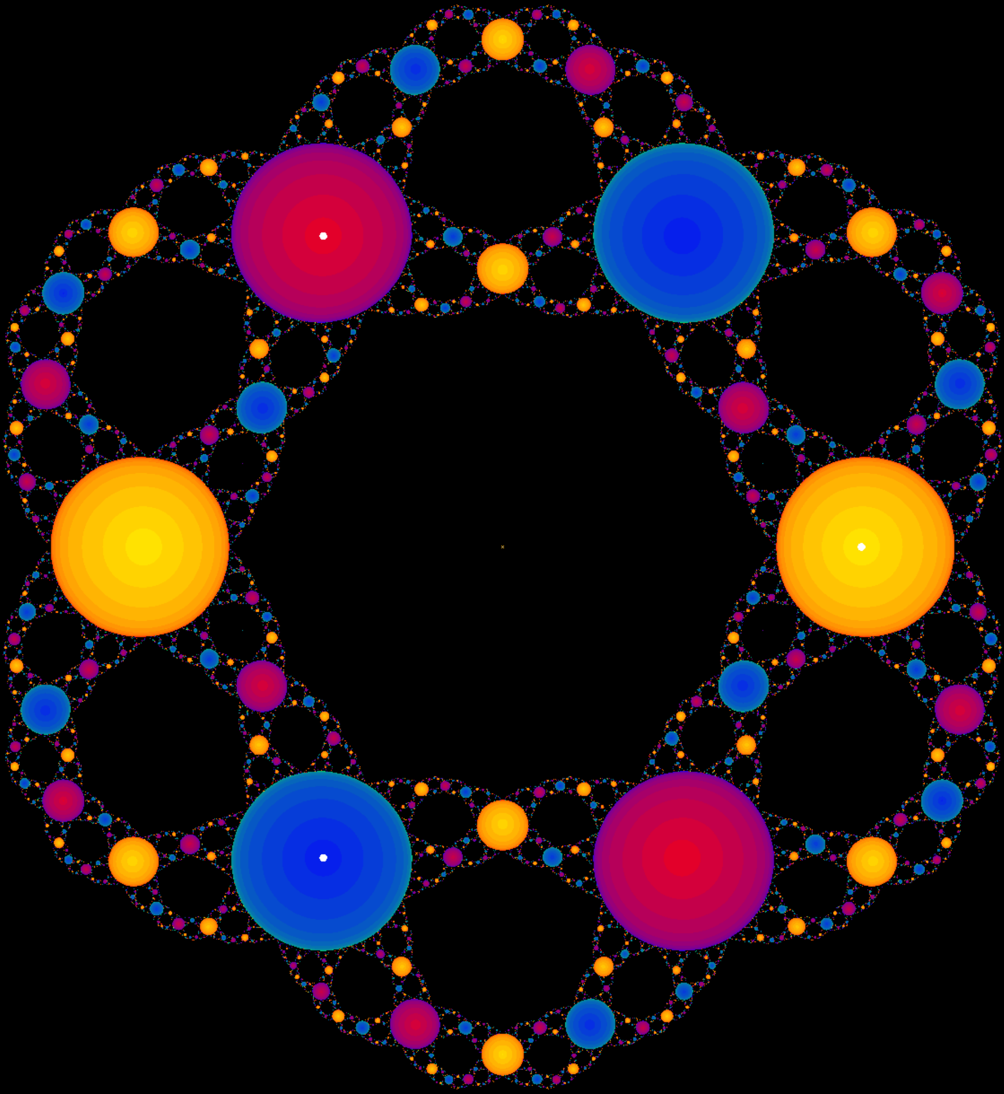


Pseudo Roots

The function $f(z) = \frac{1}{z^3 - 1} - 1$ is zero when $z^3 = 2$. In other words, it has three proper roots. On the other hand, it goes off to infinity when $z^3 = 1$. Below is a 3D plot of the modulus of the function (the colours are determined by the argument). You can see that the function has three infinite spikes at the cube roots of 1. It also dips down to zero at the cube roots of 2.

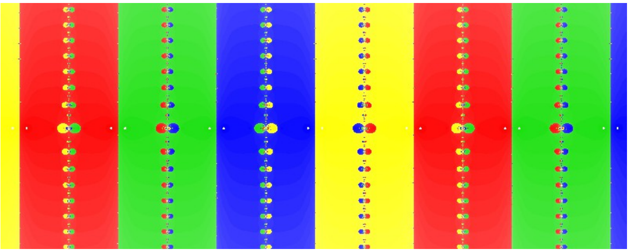


The spikes are called its pseudo-roots because a simple modification of the Newton-Raphson algorithm will search these out instead. The results are shown opposite.

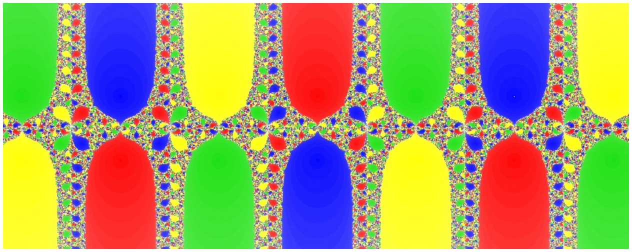


The Cosine Function

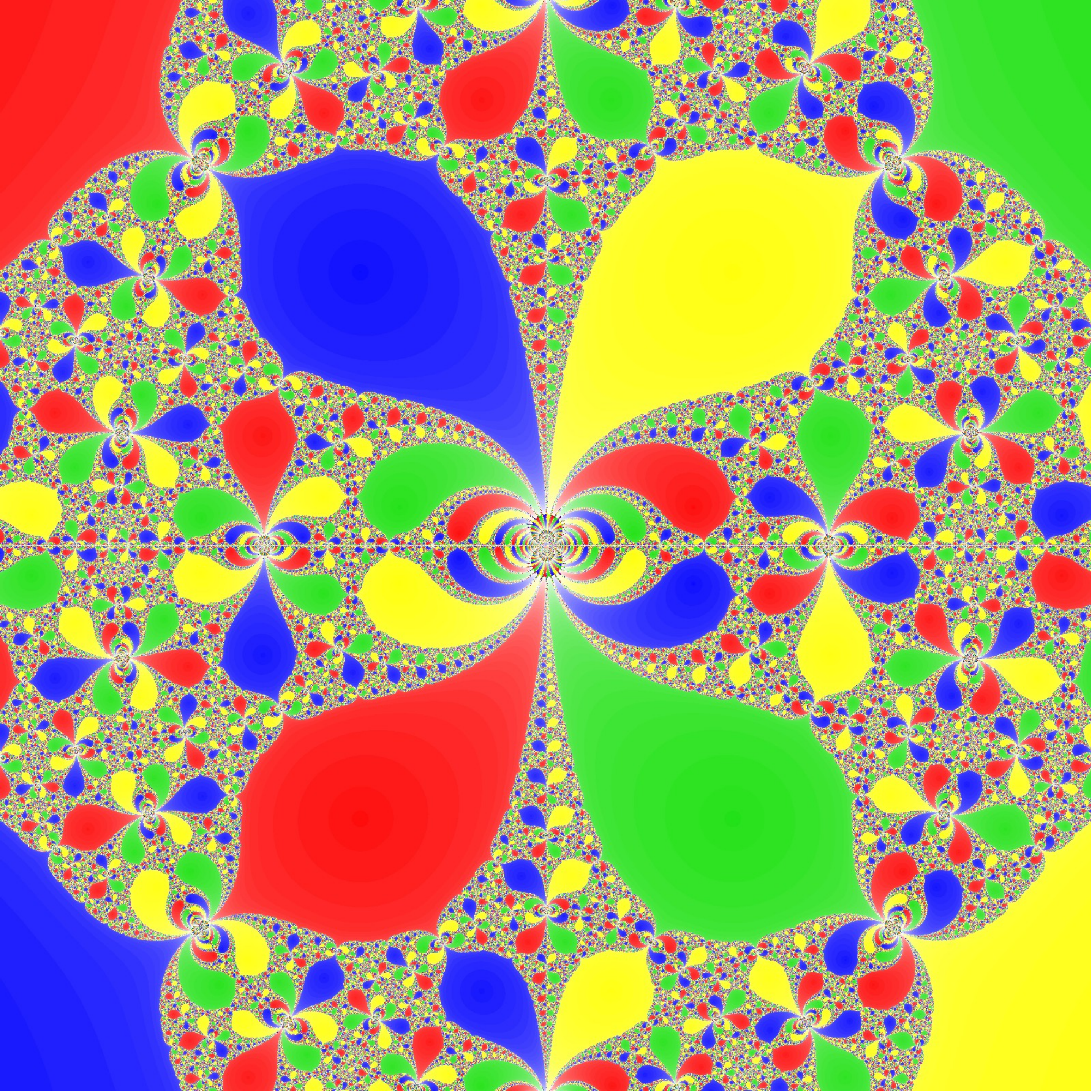
The equation $\cos(z) = k$ has multiple real roots (provided that $k <= 1$). The roots are symmetrical about the imaginary axis and are repeated at intervals of 2π on either side. Broadly speaking, the majority of starting points within each vertical band of width 2π homes in on the closest root but some points which start close to the centre line can migrate to other nearby roots. (In fact, there are points which end up at a root miles away!)



When k is greater than 1 then the pairs of roots first merge and then diverge in the imaginary direction



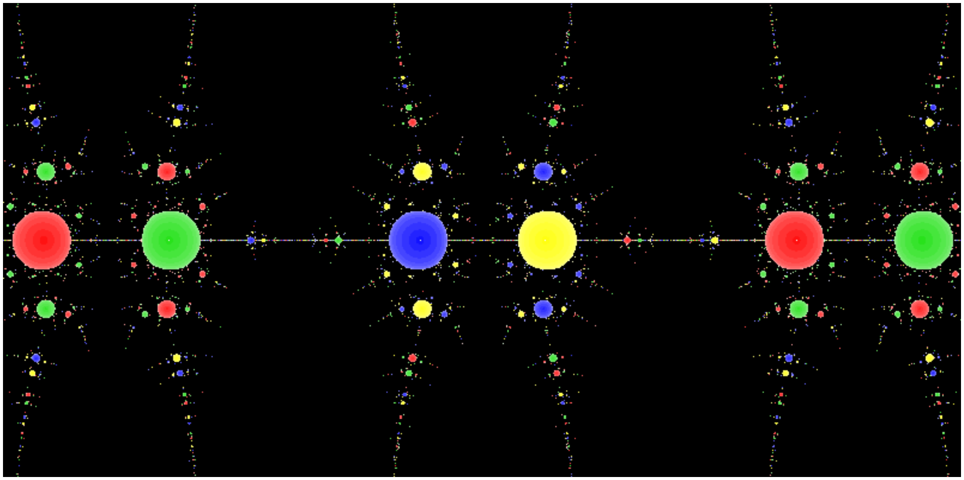
A magnified portion of the region near the origin (with $k = 4$) is shown in the illustration opposite.



The Secant Function

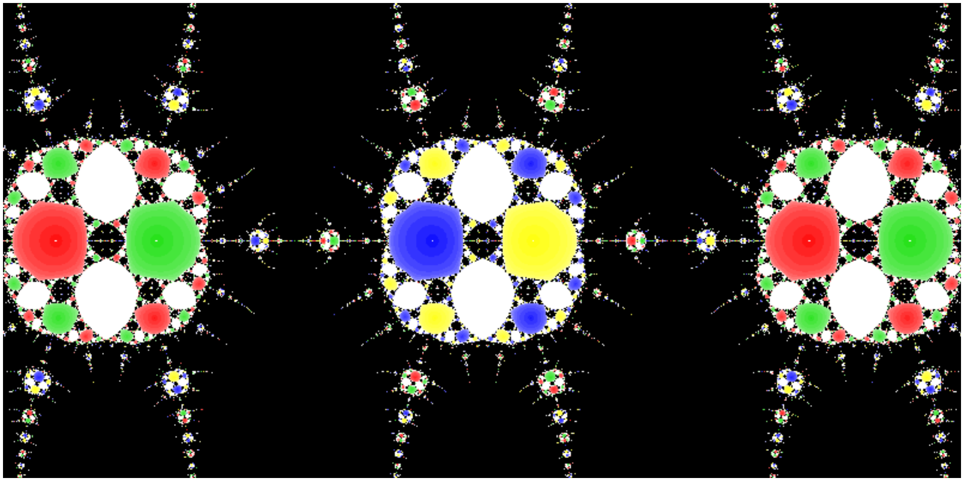
$\sec(z) = \frac{1}{\cos}(z)$ Whenever $\cos(z) = 0$, $\sec(z)$ goes off to infinity. This means that, when trying to find solutions to the equation $\sec(z) = k$, we must expect many starting points to diverge towards a pseudo-root.

When k is greater then 1, the function has a series of real roots around which are approximately circular basins of attraction as illustrated below (in which $k = 2$). The black regions are points which diverge off to infinity.

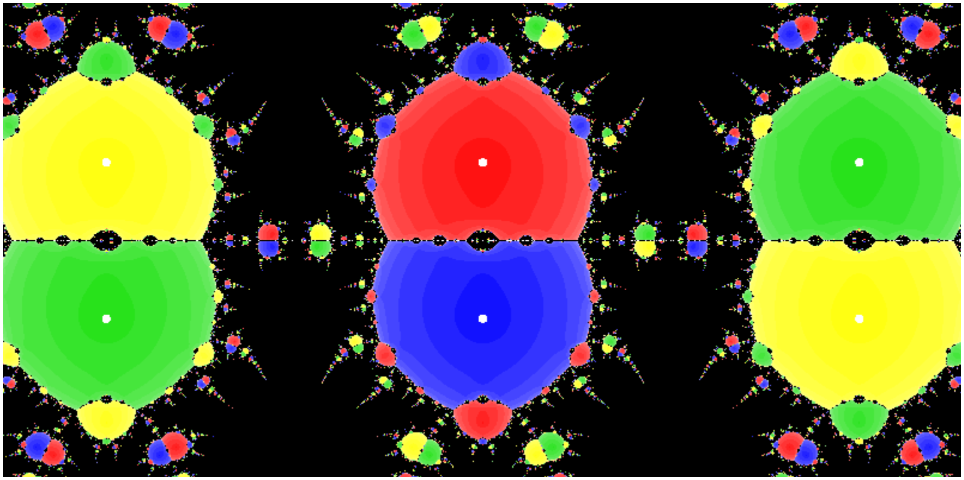


Something interesting happens when k lies between about 1.2 and 1.6. Areas of white appear in the fractal. These represent points which neither converge onto a root nor diverge to infinity. What happens is that these points converge onto a periodic repeating cycle, oscillating up and down between the

the two large approximately circular white blobs. Here $k = 1.5$

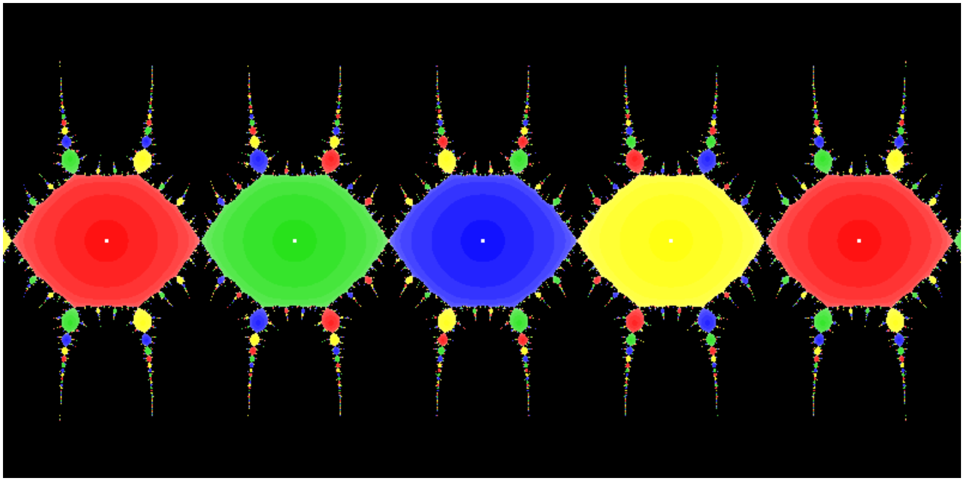


This is what happens when $k < 1$. The roots become imaginary.

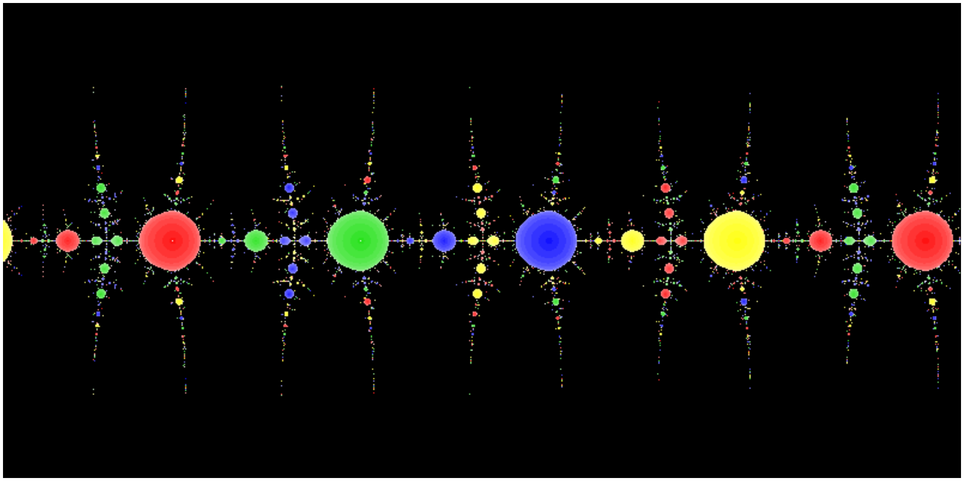


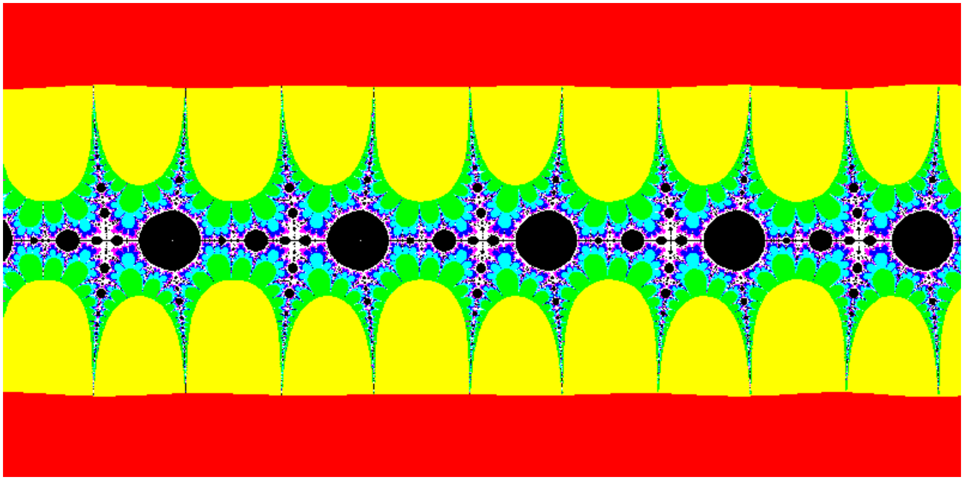
The Tangent Function

Here is the N-R fractal generated using the equation $\tan(z) = k$ with $k = 0$



As the value of k is increased, the basins of attraction get more and more distorted until somewhere between 1.5 and 1.6 the basins become separated. The next illustrations shows the fractal when $k = 2$. The upper one shows the basins of attraction while the lower one is its complement showing the speed with which the unstable points diverge.





The Exponential Function

Probably the most important complex function of them all is the exponential function e^z. If we unpack the expression we find that

$$e^z = e^{(x + iy)} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

Solutions to the equation $e^z = 1$ all have the form $x = 0$, $y = n\pi$ because $e^0 = 1$, $\cos(n\pi) = 1$ and $\sin(n\pi) = 0$. In other words, there are an infinite number of solutions at intervals of π all along the imaginary (vertical) axis.

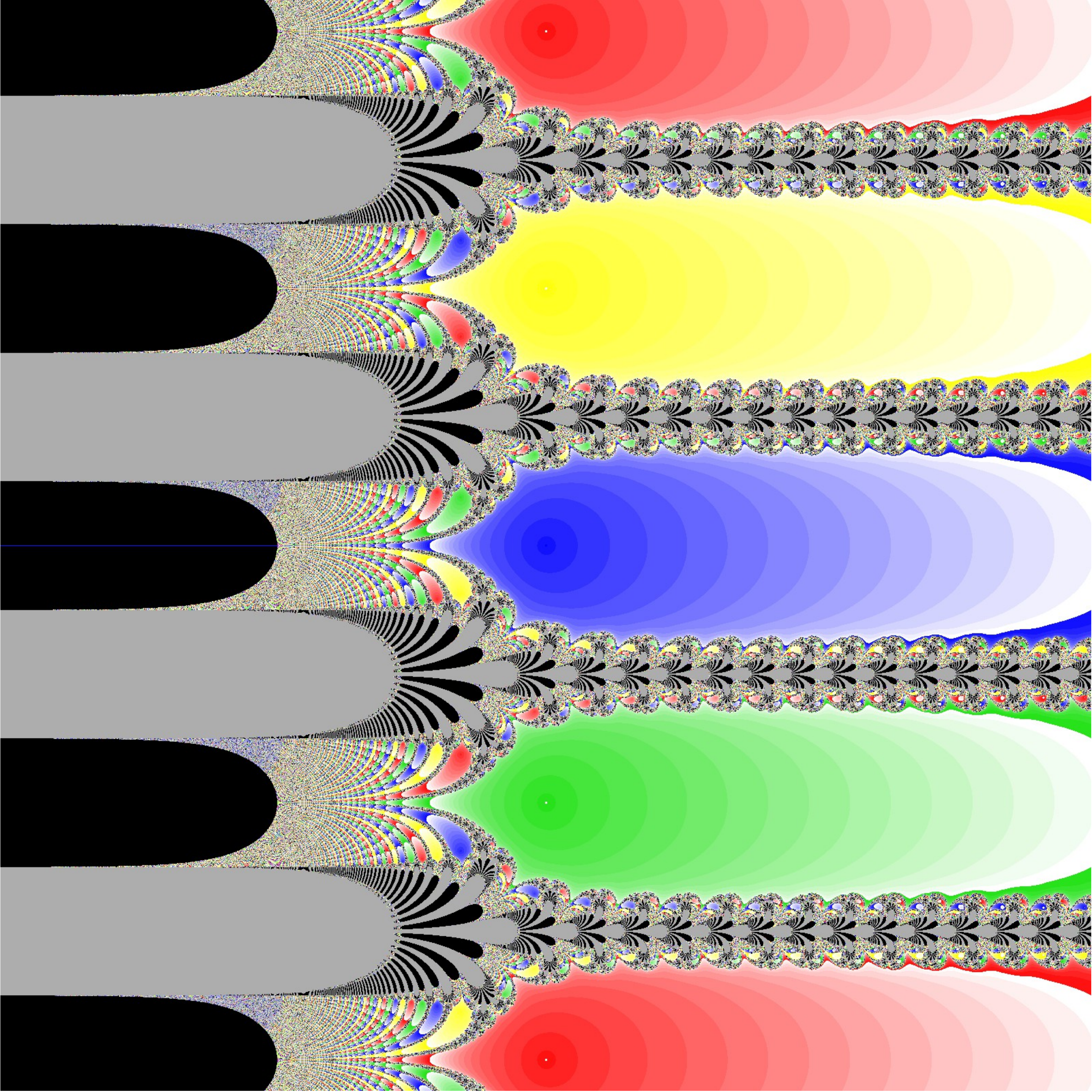
In general, starting points whose real component is positive migrate sensibly towards one of these roots but when x is negative, problems arise.

During the calculations it is necessary to divide by e^x. When x is negative, e^x is very small and so the result of the division is very large. Two things can now happen. Either the value is so large that it results in a math overflow error or the reciprocal of e^x is so small that the correcting factor is effectively zero and z approaches the solution one unit at a time!

In the image opposite the black areas are points which cause a mathematical overflow (using standard double precision arithmetic). These points should also be filled with colour but they lead to roots which will be literally miles away!

The grey areas contain those points which approach their roots incredibly slowly.

The final illustration in the book shows a magnified section (rotated by 90°.



All the images in this book were generated using a program written by the author called Mandelbrot Explorer.

The program runs exclusively under Windows and may be downloaded free from the author's website:

www.jolinton.co.uk

The author would be happy to hear from any readers who have enjoyed this book or discovered new iterative fractals themselves. The author's email is:

joliverlinton@gmail.com